The Maximum-Skew Constraint

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Abstract

In the wings of an implied-vol smile, no-butterfly-arbitrage is more than a floor on the implied PDF (or on the convexity): it is a cap on the skew. Concretely, the no-butterfly-arbitrage condition at any point reduces to a quadratic in the skew s. Where this quadratic admits two real roots $s_- < s_+$ in the regime $d_1 d_2 > 1$, the cap reads $s \leq s_-$ at any right-wing point, with mirror $s \geq s_+$ at the left; we call this the *maximum-skew constraint*. Under strict Lee [3], this regime holds asymptotically in either wing. This note studies that quadratic, first in the linear-in-variance edge-knot case introduced by the CVI paper [1], then in full generality.

At the edge knot of CVI's cubic spline, linear-in-variance wing extrapolation forces $c = 0$. The CVI paper selected the smaller-magnitude root on practical grounds, leaving open whether any strike-arbitrage-free smile could lie on the larger-magnitude root. We close that gap via Lucic's pointwise strike-arbitrage slope bound [6], which sits strictly between s_- and s_+: the larger-magnitude root carries a vertical-spread arbitrage incompatible with strike-arbitrage-freeness. We further identify the constraint as the *finite-strike form of Lee's asymptotic large-strike bound* [3]: strictly tighter than Lee at any finite knot, approaching Lee only as the knot moves deep into the tail, and propagating along the linear-extrapolated wing to give necessary-and-sufficient strike-arbitrage-freeness *at and beyond* the knot. Furthermore, an equivalent form free of the $0/0$ pathology at $d_1 d_2 = 1$ is given. The argument applies to any implied-volatility parametrization with linear-in-variance wing extrapolation.

Beyond CVI, the $c = 0$ assumption is relaxed. Strict Lee gives the wing regime asymptotically; for asymptotically linear smooth tails with $c(z) \to 0$, including SVI/SSVI, the remaining real-root condition $c \leq c^*(z)$ also holds asymptotically in either wing. The larger-magnitude root is excluded wherever such roots exist. Under Black-Scholes pricing, $s \leq s_-$ alone is pointwise (i.e., at each finite strike of the slice) necessary and sufficient for strike-arbitrage-freeness (mirror $s \geq s_+$ at the left); the name *maximum-skew constraint* thus becomes literal: in either wing, wherever the smaller-magnitude root exists, it is the maximum admissible skew. Without Black-Scholes, the combined form $s \leq \min\{s_-,\ s_{\mathrm{VS}}^{\mathrm{call}}\}$ takes over, with butterfly binding for $c < c_{\mathrm{cross}}$ and the call-spread Mills threshold for $c > c_{\mathrm{cross}}$, where c_{cross} is the convexity at which s_- and $s_{\mathrm{VS}}^{\mathrm{call}}$ coincide. Moving out along an asymptotic linear-in-variance wing, $c_{\mathrm{cross}}(z) \sim 1/z$; under the further assumption of eventually monotone convexity, $c(z) = o(1/z)$, so the Mills-ratio bound is asymptotically non-binding. For parametrizations with positive fast-decaying convexity ($c(z) \geq 0$ eventually and $c(z) = o(1/z^2)$; covering SVI and SSVI with $c(z) = O(1/z^3)$), the pointwise picture above extends asymptotically to a sufficient (not necessary) at-and-beyond form: a single-strike check against the CVI $c = 0$ bound $s_-(z_b, 0)$ (sufficient since $c \geq 0$ implies $s_-(z_b, 0) \leq s_-(z_b, c)$) propagates strike arbitrage-freeness along the entire wing past the strike, with the closed form worked out for raw SVI.

Across the smile, the picture extends into the interior as a band $s_+ \leq s \leq s_-$ on the skew when butterfly roots are real (automatic for any $c \geq 0$, since the interior c^* is negative). At very marked W shapes near the money, the butterfly quadratic has no real roots and the constraint becomes a convexity floor $c \geq c^*$. In short, butterfly arbitrage is best read as a cap or floor on the skew across the smile when real roots exist.

"Nature abhors a naked singularity."
Stephen Hawking

1 Setup and the edge butterfly quadratic

Let us first fix notation and recall the edge-knot butterfly quadratic from [1].

Notation. Following [1], we let $k := \log(K/F)$ be the log-forward moneyness, $v(k) := \sigma(k)^2$ the implied-variance curve at a fixed expiry T, and $\sigma_\star$ the anchor ATM volatility (a pre-fit estimate, fixed before calibration). We write $v_\star := \sigma_\star^2$ and $w_\star := v_\star T$, and introduce the normalized log-moneyness $z := k/\sqrt{w_\star}$. The shape function is $\psi(z) := v/v_\star$, and we write $s := \partial_z\psi$, $c := \partial_z^2\psi$ for its first two derivatives.

The CVI parameterization places cubic-spline knots $z_0 < z_1 < \cdots < z_{n-1}$ in normalized moneyness, with linear extrapolation beyond z_0 and z_{n-1}. This forces $c = 0$ at both edges, which is what makes the butterfly condition at those points take the simple quadratic form below.

Strict-Lee assumption. Throughout, we assume the strict form of Lee's tail-slope bound [3]: $\beta := \limsup_{k\to\infty} w(k)/k < 2$ (mirror at the left wing by $k \to -k$ symmetry). Equivalently, the underlying admits some moment of order strictly greater than 1.

Terminology: pointwise versus at-and-beyond. Throughout, "pointwise" means that the stated skew constraint is the local condition to impose at a given finite strike, within a slice whose no-butterfly condition is enforced at every finite strike. This is contrasted with the "at-and-beyond" results, where a single check at a boundary strike propagates along the extrapolated wing.

The edge butterfly quadratic and its admissible regime. At any point with $c = 0$, the no-butterfly-arbitrage condition of Martini and Mingone [2], in CVI notation, reads:

$$v_\star\left(\frac{d_1 d_2 - 1}{2v}\right)s^2 + \sqrt{v_\star}\left(\frac{2d_1}{\sqrt{v}} - \sqrt{T}\right)s + 2 \;\geq\; 0, \tag{1}$$

with $d_{1,2} = (-k \pm vT/2)/\sqrt{vT}$. Strict Lee, under linear-variance wings, guarantees $d_1 d_2 > 1$ for all $|k|$ sufficiently large (verified in §4, "Relation to Lee's bounds"); we therefore restrict attention throughout to edge knots placed deep enough into the tails that $d_1 d_2 > 1$, i.e., $|k| > \sqrt{vT}\sqrt{1 + vT/4}$. We refer to this admissibility regime as "the wing" throughout (right or left according to the sign of k), and use "in the wing" and "$d_1 d_2 > 1$" interchangeably (pointwise at k).

In this regime the quadratic (1) has two real roots $s_- < s_+$, and the butterfly-feasible region is $s \le s_-$ or $s \ge s_+$. The two roots behave as follows at the edges:

- At the *right* edge z_{n-1}: both roots are positive, so $0 < s_- < s_+$. The root s_- is the one closer to zero.
- At the *left* edge z_0: both roots are negative. The ordering $s_- < s_+$ is preserved, so s_- is the more-negative root and s_+ the less-negative, closer-to-zero one.

The CVI paper [1] selected the root closer to zero at each edge: $s \le s_-$ at z_{n-1} and $s \ge s_+$ at z_0. The opposite root was rejected on the practical grounds that, even at the retained root, the implied skew is already at the limit of strike arbitrage; the opposite root would be steeper still.

What this note adds for CVI. This note gives a theoretical justification for that selection rule. We work at the right edge z_{n-1} throughout; the corresponding statements at z_0 follow by the $k \to -k$ symmetry, which flips the sign of the linear coefficient of (1) (using $d_1 - d_2 = \sqrt{vT}$, this coefficient simplifies to $-2k\sqrt{v_\star}/(v\sqrt{T})$) and of both roots. The selection rule has a single unified form: at each edge, the excluded branch is the root of larger magnitude (positive s_+ at the right, more-negative s_- at the left). We further show that this constraint is the finite-strike form of Lee's asymptotic large-strike bound at the edge knot. In practice, the asymptotic Lee bound is known to be too loose for calibration on traded smiles [10]; the present work tightens it rigorously.

Beyond CVI. Although stated in CVI variables for concreteness, the argument applies to any implied-volatility parametrization with linear-in-variance extrapolation at the wing. §8 is broader still: its pointwise statement drops the linear-in-variance assumption and extends the larger-magnitude-root exclusion to general convexity c. Under Black-Scholes pricing, $s \le s_-$ is pointwise necessary and sufficient at any right-wing point where s_- exists (mirror at the left); without Black-Scholes, the call-spread Mills threshold also enters. §10 closes the note with the deep-wing asymptotics and works out SVI and SSVI explicitly: a single-strike sufficient check for strike arbitrage-freeness on the entire wing past the chosen strike, applying more generally to any parametrization with asymptotic linear-in-variance wing, $c(z) \ge 0$ eventually, and $c(z) = o(1/z^2)$. Strict Lee enters as a separate parametric input for SVI/SSVI ($b(1+\rho) < 2$ for SVI; analogous for SSVI), whereas the CVI case derives it from the boundary check.

2 The edge quadratic in normalized form

Let us rewrite the discriminant and the roots of the edge butterfly quadratic (1) in the normalized variables of §1. The derivations below evaluate at a generic point where $c = 0$; with $\psi := v/v_\star$ at that point, we have $k = z\sqrt{w_\star}$ and $vT = w_\star\psi$.

Discriminant. The discriminant Δ of (1) as a quadratic in s is given by

$$\Delta = v_\star\left[\left(\frac{2d_1}{\sqrt{v}} - \sqrt{T}\right)^2 - \frac{4(d_1d_2 - 1)}{v}\right].$$

Using the relationship $d_1 - d_2 = \sqrt{vT}$, we first rewrite the leading term as $2d_1/\sqrt{v} - \sqrt{T} = (d_1 + d_2)/\sqrt{v}$. The square bracket then reads $(d_1 + d_2)^2/v - 4(d_1d_2 - 1)/v$, and applying the identity $(a+b)^2 - 4ab = (a-b)^2$ to (d_1, d_2) gives $[(d_1 - d_2)^2 + 4]/v = T + 4/v$. Hence:

$$\Delta \;=\; v_\star\left(T + \frac{4}{v}\right) \;=\; w_\star + \frac{4}{\psi}. \tag{2}$$

Notably, Δ depends on k only through $v(k)$; the log-moneyness k does not enter directly. This is inherited from $d_1 - d_2 = \sqrt{vT}$, which is itself k-free: the k-terms cancel inside the bracket above, leaving only T and $4/v$.

Roots. Solving the quadratic for s, when $d_1d_2 > 1$, we obtain the two roots

$$s_\pm \;:=\; \frac{2z \pm \psi\sqrt{\Delta}}{d_1d_2 - 1}. \tag{3}$$

Vieta's product-of-roots identity. It is convenient to record, at this stage, an identity that will be used repeatedly in what follows. Combining (2) with the identity $d_1d_2 = z^2/\psi - w_\star\psi/4$, we obtain

$$4\psi\,(d_1d_2 - 1) \;=\; 4z^2 - \psi^2\Delta. \tag{4}$$

Equivalently, this is Vieta's product-of-roots applied to (1): $s_+s_- = C/A = 4\psi/(d_1d_2-1)$, matched against $(4z^2-\psi^2\Delta)/(d_1d_2-1)^2$ from (3). In particular, $d_1d_2>1$ is equivalent to $4z^2>\psi^2\Delta$, or $|2z|>\psi\sqrt{\Delta}$.

3 Lucic's slope bound in CVI variables

The pointwise slope bound on implied vol arising from vertical-spread no-arbitrage goes back to Hodges [7]. We use the normalizing-volatility-transform formulation of Lucic [6] and translate it into the normalized CVI variables of §1. This is the input that will let us separate the two butterfly roots in §4.

Lucic's bound in its original form. Lucic [6] established, for any differentiable strike-arbitrage-free slice (with total implied variance $w = vT$ in the CVI convention), a slope bound (his Proposition 3.5) on the total implied vol $\sqrt{w}$, in two one-sided forms. For $k > k_1^* := vT/2$, the zero of the first normalizing-volatility transform $f_1 := k/\sqrt{w} - \sqrt{w}/2 = -d_1$, the bound reads

$$\partial_k \sqrt{w(k)} \;<\; \frac{2\sqrt{w(k)}}{2k + w(k)}. \tag{5}$$

By Lucic's companion bound past the zero $k_2^* := -vT/2$ of $f_2 := k/\sqrt{w} + \sqrt{w}/2 = -d_2$, the mirror bound for $k < k_2^*$ reads

$$\partial_k \sqrt{w(k)} \;>\; \frac{2\sqrt{w(k)}}{2k - w(k)}.$$

Validity at the edges. At any edge knot the regime condition $d_1d_2 > 1$ forces d_1 and d_2 to share a sign. At the right edge $k > 0$ makes $d_2 < 0$, so $d_1 < 0$ too and $k > vT/2 = k_1^*$; the f_1-side bound (5) applies. At the left edge $k < 0$ makes $d_1 > 0$, so $d_2 > 0$ too and $k < -vT/2 = k_2^*$; the f_2-side mirror bound applies.

Translation to CVI variables. Let us convert (5) into the normalized variables of §1. With $w = vT = w_\star\psi$ and $k = z\sqrt{w_\star}$, we have $\sqrt{w} = \sqrt{w_\star\psi}$. Applying the chain rule to $\sqrt{w(k)} = \sqrt{w_\star\,\psi(z(k))}$, using $\partial_z\psi = s$ and $\partial_k z = 1/\sqrt{w_\star}$,

$$\partial_k\sqrt{w} \;=\; \frac{\sqrt{w_\star}}{2\sqrt{\psi}}\,\partial_k\psi \;=\; \frac{\sqrt{w_\star}}{2\sqrt{\psi}}\cdot\frac{s}{\sqrt{w_\star}} \;=\; \frac{s}{2\sqrt{\psi}}.$$

On the right-hand side of (5), with $2k + w = \sqrt{w_\star}(2z + \sqrt{w_\star}\,\psi)$ and $\sqrt{w} = \sqrt{w_\star\,\psi}$,

$$\frac{2\sqrt{w}}{2k+w} \;=\; \frac{2\sqrt{w_\star\,\psi}}{\sqrt{w_\star}\,(2z+\sqrt{w_\star}\,\psi)} \;=\; \frac{2\sqrt{\psi}}{2z+\sqrt{w_\star}\,\psi}.$$

Putting the two sides together and clearing the factor $1/(2\sqrt{\psi})$, the Lucic bound (5) becomes a direct upper bound on the normalized skew at the right edge, which we denote s_L^{right}:

$$s \;<\; s_L^{\text{right}} \;:=\; \frac{4\psi}{2z+\sqrt{w_\star}\,\psi}. \tag{6}$$

Mirror bound at the left edge. The same computation, applied to the f_2-side bound (or equivalently obtained by $k \to -k$ symmetry), yields

$$s \;>\; s_L^{\text{left}} \;:=\; \frac{4\psi}{2z-\sqrt{w_\star}\,\psi} \qquad \text{at the left edge.}$$

The two bounds have opposite signs, directly from the sign of their denominators: at $z_{n-1} > 0$ the denominator $2z + \sqrt{w_\star}\,\psi$ is a sum of two positive terms, so $s_L^{\text{right}} > 0$; at $z_0 < 0$ the denominator $2z - \sqrt{w_\star}\,\psi$ is a sum of two negative terms, so $s_L^{\text{left}} < 0$.

The translation above is purely a change of variables; it carries Lucic's strike-arbitrage-freeness hypothesis unchanged.

Where Lucic's bound sits in the strike-arbitrage stack. The bound (5) is a necessary (but not sufficient) condition for absence of vertical-spread arbitrage [6]; we use it as a one-way exclusion test at the edge knot (violation by s_+ rules out the upper root). The CVI paper [1] does not separately enforce vertical-spread: as elaborated in §5, it is implied by absence of butterfly arbitrage, strict Lee, and the Black-Scholes intrinsic-value lower bound. The proof-by-contradiction in §5 establishing infeasibility of s_+ relies on this same chain.

Scope of the remainder. For the rest of this note we specialize to the right edge: z and ψ hereafter denote z_{n-1} and $\psi(z_{n-1})$ respectively. The mirror at the left edge is stated as a paragraph after the derivation; §8 relaxes the edge-knot specialization to any right-wing point, and §9 treats both edges explicitly.

4 Lucic's bound between the butterfly roots

Recall that z_0 and z_{n-1} are the edge knots of the CVI cubic spline (leftmost and rightmost in normalized log-moneyness), and that at both of them linear extrapolation forces $c = 0$, reducing the no-butterfly-arbitrage condition of Martini and Mingone to the quadratic (1) in s. The derivation below is at the right edge $z_{n-1} > 0$; the mirror statement at $z_0 < 0$ is recorded immediately after.

Let us now position the Lucic bound s_L^{right} relative to the two butterfly roots $s_\pm$ at the right edge knot. We show that s_L^{right} lies strictly between them.

Upper-root side. At the right edge, the denominator of $s_L^{\text{right}} = 4\psi/(2z + \sqrt{w_\star}\,\psi)$ is positive (sum of two positives), so multiplying through by it preserves the inequality direction:

$$s_+ > s_L^{\text{right}} \iff (2z + \sqrt{w_\star}\,\psi)\, s_+ - 4\psi > 0.$$

Let us compute the left-hand side directly. Using the explicit form (3) of s_+ and clearing its denominator $d_1 d_2 - 1$,

$$(2z + \sqrt{w_\star}\,\psi)\, s_+ - 4\psi \;=\; \frac{(2z + \sqrt{w_\star}\,\psi)(2z + \psi\sqrt{\Delta}) - 4\psi(d_1 d_2 - 1)}{d_1 d_2 - 1}.$$

By Vieta's identity (4), the subtracted term in the numerator equals $4z^2 - \psi^2\Delta$. Expanding the product and combining,

$$\begin{aligned}(2z + \sqrt{w_\star}\,\psi)(2z + \psi\sqrt{\Delta}) - (4z^2 - \psi^2\Delta) &= 4z^2 + 2z\psi\sqrt{\Delta} + 2z\sqrt{w_\star}\,\psi + \sqrt{w_\star}\,\psi^2\sqrt{\Delta} - 4z^2 + \psi^2\Delta \\ &= 2z\psi\sqrt{\Delta} + 2z\sqrt{w_\star}\,\psi + \sqrt{w_\star}\,\psi^2\sqrt{\Delta} + \psi^2\Delta.\end{aligned}$$

Grouping by the common factor $\sqrt{\Delta} + \sqrt{w_\star}$,

$$\begin{aligned}2z\psi\sqrt{\Delta} + 2z\sqrt{w_\star}\,\psi + \sqrt{w_\star}\,\psi^2\sqrt{\Delta} + \psi^2\Delta &= \psi\big[2z(\sqrt{\Delta} + \sqrt{w_\star}) + \psi\sqrt{\Delta}(\sqrt{\Delta} + \sqrt{w_\star})\big] \\ &= \psi(\sqrt{\Delta} + \sqrt{w_\star})(2z + \psi\sqrt{\Delta}).\end{aligned}$$

Hence

$$(2z + \sqrt{w_\star}\,\psi)\, s_+ - 4\psi \;=\; \frac{\psi(\sqrt{\Delta} + \sqrt{w_\star})(2z + \psi\sqrt{\Delta})}{d_1 d_2 - 1}.$$

Consequently, since $z, \psi, \sqrt{\Delta}, \sqrt{w_\star}$ and $d_1 d_2 - 1$ are all strictly positive at the right edge, the right-hand side is positive. Dividing through by the positive factor $2z + \sqrt{w_\star}\,\psi$, we obtain $s_+ > s_L^{\text{right}}$. The upper root therefore violates Lucic's bound (6).

Lower-root side. The analogous computation for s_- yields

$$(2z + \sqrt{w_\star}\,\psi)\, s_- - 4\psi \;=\; \frac{-\psi(\sqrt{\Delta} - \sqrt{w_\star})(2z - \psi\sqrt{\Delta})}{d_1 d_2 - 1}.$$

The factor $\sqrt{\Delta} - \sqrt{w_\star}$ is positive since $\Delta = w_\star + 4/\psi > w_\star$. It remains to check the sign of $2z - \psi\sqrt{\Delta}$. At the right edge both z and $\psi\sqrt{\Delta}$ are positive, so we may square both sides:

$$2z - \psi\sqrt{\Delta} \;>\; 0 \iff 4z^2 \;>\; \psi^2\Delta,$$

and by Vieta's identity (4) this last inequality is exactly the regime condition $d_1 d_2 > 1$. Hence both factors $(\sqrt{\Delta} - \sqrt{w_\star})$ and $(2z - \psi\sqrt{\Delta})$ are positive; together with the leading minus sign the numerator is negative, and the whole expression is negative. Dividing through by the positive factor $2z + \sqrt{w_\star}\,\psi$, we obtain $s_- < s_L^{\text{right}}$.

Combined inequality. Combining the two sides,

$$\boxed{s_- \;<\; s_L^{\text{right}} \;<\; s_+\,,} \tag{7}$$

when $d_1 d_2 > 1$. Consequently, the upper root s_+ violates Lucic's strike-arbitrage-free slope bound and is therefore inadmissible for the edge slice. Figure 1 visualizes the interlacing at flat vol.

Refinement via the Mills-ratio bound. Vertical-spread arbitrage has two sides: *call-spread arbitrage* ($\partial_K C > 0$) and *put-spread arbitrage* ($\partial_K P < 0$). Lucic's Proposition 3.5 has two one-sided forms (already given in §3): the right-wing slope bound (5) is necessary for absence of call-spread arbitrage; the left-wing mirror is necessary for absence of put-spread arbitrage; each is necessary but not sufficient for its side. At $s = s_+$ on the right edge the active failure is on the call-spread side (see §5); we restrict to that side from here on. The tighter Mills-ratio bound (Lemma 3.7 of [6], stated there as necessary for strike-arbitrage-freeness) is in fact pointwise necessary *and sufficient* for absence of call-spread arbitrage: $\partial_k\sqrt{w} \le R(f_2)$, where $R(t)$ denotes the standard-normal Mills ratio (the probability that the standard normal exceeds t, divided by the density at t). The biconditional follows from [6, eq. (2.7)] together with $\partial_K C = P(K) - 1$: $\partial_k\sqrt{w} \le R(f_2) \iff P(K) \le 1 \iff \partial_K C \le 0$. The put-spread mirror $\partial_k\sqrt{w} \ge -R(-f_2)$, derived analogously from $P(K) \ge 0$ in [6, eq. (2.7)], is not invoked here. The Mills-ratio bound yields the strict refinement $s_- < s_{\text{VS}}^{\text{call}} < s_L^{\text{right}} < s_+$ when $d_1 d_2 > 1$, where $s_{\text{VS}}^{\text{call}}$ is the call-spread Mills-ratio threshold in CVI variables.

Proofs of the two strict inequalities. The middle inequality $s_{\text{VS}}^{\text{call}} < s_L^{\text{right}}$ follows from Gordon's upper bound [8] $R(t) < 1/t$ for $t > 0$ at the right edge, where $f_2 > 0$. The lower-root inequality $s_- < s_{\text{VS}}^{\text{call}}$ follows from Gordon's lower bound $R(t) > t/(t^2+1)$ for $t > 0$: this gives $s_{\text{VS}}^{\text{call}} > 2\sqrt{\psi}\, f_2/(f_2^2 + 1) = (2z + \psi\sqrt{w_\star})/(f_2^2 + 1)$. Cross-multiplying $s_- = 4\psi/(2z + \psi\sqrt{\Delta}) < (2z + \psi\sqrt{w_\star})/(f_2^2 + 1)$ with both denominators positive, and using $\psi^2\Delta = w_\star\psi^2 + 4\psi$ to simplify $4\psi(f_2^2+1) = 4z^2 + 4z\psi\sqrt{w_\star} + \psi^2\Delta$, the difference $(2z+\psi\sqrt{\Delta})(2z+\psi\sqrt{w_\star}) - 4\psi(f_2^2+1)$ factors as

$\psi(\sqrt{\Delta}-\sqrt{w_\star})(2z-\psi\sqrt{\Delta})$ with the first two factors strictly positive ($\Delta > w_\star$). The remaining inequality $\psi\sqrt{\Delta} < 2z$ is equivalent to $d_1 d_2 > 1$ by (4).

The sequence of arbitrages encountered as the skew rises at the edge knot is: past s_-, butterfly arbitrage appears; past $s_{\mathrm{VS}}^{\mathrm{call}}$, call-spread arbitrage joins on top of it; past s_+, butterfly clears and only call-spread remains. Crossing s_L^{right} does not change the arbitrage state, since Lucic's bound is a strict relaxation of the Mills-ratio bound and no new failure appears there. All three thresholds share the same leading-order deep-wing limit $\sqrt{w_\star}\, s \to 4w/(2k+w) = 4\beta/(2+\beta)$ in the linear-wing regime $w(k) \sim \beta k$ (butterfly via $\sqrt{w(w+4)} \to w$, Mills via Gordon's $f_2 R(f_2) \to 1$, Lucic by construction); the gaps in this finite-wing ordering shrink at rate $O(1/k)$ in the deep wing (computed in §10).

The inequality $s_+ > s_L^{\mathrm{right}}$ established in (7) is used in the larger-magnitude-root exclusion of §5. The Mills-level chain $s_+ > s_{\mathrm{VS}}^{\mathrm{call}}$ (Gordon's upper bound) and Mills sufficiency (Lemma 3.7 of [6]) are used in the pointwise generalization to $c \neq 0$ of §8.

Mirror statement at the left edge. By the $k \to -k$ symmetry of §1, each step of the derivation above has an exact mirror at z_0. The resulting inequality is

$$s_- \;<\; s_L^{\mathrm{left}} \;<\; s_+.$$

There the excluded root is the *lower* one s_-, the more-negative; the retained root is the upper one s_+, the less-negative, closer-to-zero. At either edge the excluded root is the one of larger magnitude; at the right edge this is s_+, at the left edge s_- (the two roots cross zero between edges while preserving the ordering $s_- < s_+$).

Relation to Lee's bounds. Let us compute how $s_\pm$ behaves asymptotically under linear extrapolation, to compare against Lee's tail-slope bound. Under linear-in-variance extrapolation the strict-Lee $\limsup$ of §1 is realized as a limit: $w(k) \sim \beta k$ as $k \to \infty$ with $\beta < 2$. We further assume $\beta > 0$ (so $\beta \in \,]0,2[$); the degenerate $\beta = 0$ case has bounded ψ and gives the trivial limits $s_\pm \to 0$, of no interest here.

In this regime $\psi = w/w_\star \sim \beta k/w_\star$. The product $d_1 d_2 = z^2/\psi - w_\star\psi/4$ therefore behaves as

$$d_1 d_2 \;\sim\; \frac{k}{\beta} - \frac{\beta k}{4} \;=\; \frac{k(4-\beta^2)}{4\beta}. \tag{8}$$

Since $\beta \in \,]0,2[$, the coefficient $(4-\beta^2)/(4\beta)$ is strictly positive; hence $d_1 d_2 \to +\infty$ as $k \to \infty$, and in particular $d_1 d_2 > 1$ for all $|k|$ sufficiently large. In other words, *linear variance tails together with strict Lee imply $d_1 d_2 > 1$ asymptotically*; this is the regime condition invoked (without explicit derivation) in [1] and used throughout this note.

Since $\beta > 0$ gives $\psi \to \infty$ in the tail, $\Delta = w_\star + 4/\psi \to w_\star$, and the numerator of (3) behaves as $2z \pm \psi\sqrt{\Delta} \sim 2z \pm \psi\sqrt{w_\star} \sim (2\pm\beta)\,k/\sqrt{w_\star}$. Substituting into (3) and taking the limit,

$$\sqrt{w_\star}\, s_+ \;\to\; \frac{4\beta}{2-\beta}, \qquad \sqrt{w_\star}\, s_- \;\to\; \frac{4\beta}{2+\beta}.$$

The strict-Lee assumption $\beta < 2$ above rewrites in CVI variables as $\sqrt{w_\star}\, s < 2$ at the right wing. The upper-root limit $4\beta/(2-\beta)$ is strictly less than 2 whenever $\beta < 2/3$: Lee's bound alone therefore does *not* rule out the upper root for wing slopes in that range. Lucic's bound (6) does.

By $k \to -k$ symmetry the same picture holds at the left wing, with signs flipped: $\sqrt{w_\star}\, s_+ \to -4\beta/(2+\beta)$ (smaller magnitude) and $\sqrt{w_\star}\, s_- \to -4\beta/(2-\beta)$ (larger magnitude); the strict-Lee bound becomes $\sqrt{w_\star}\, s > -2$, leaving the excluded lower root s_- admissible to Lee for $\beta < 2/3$ and ruled out by Lucic's mirror bound.

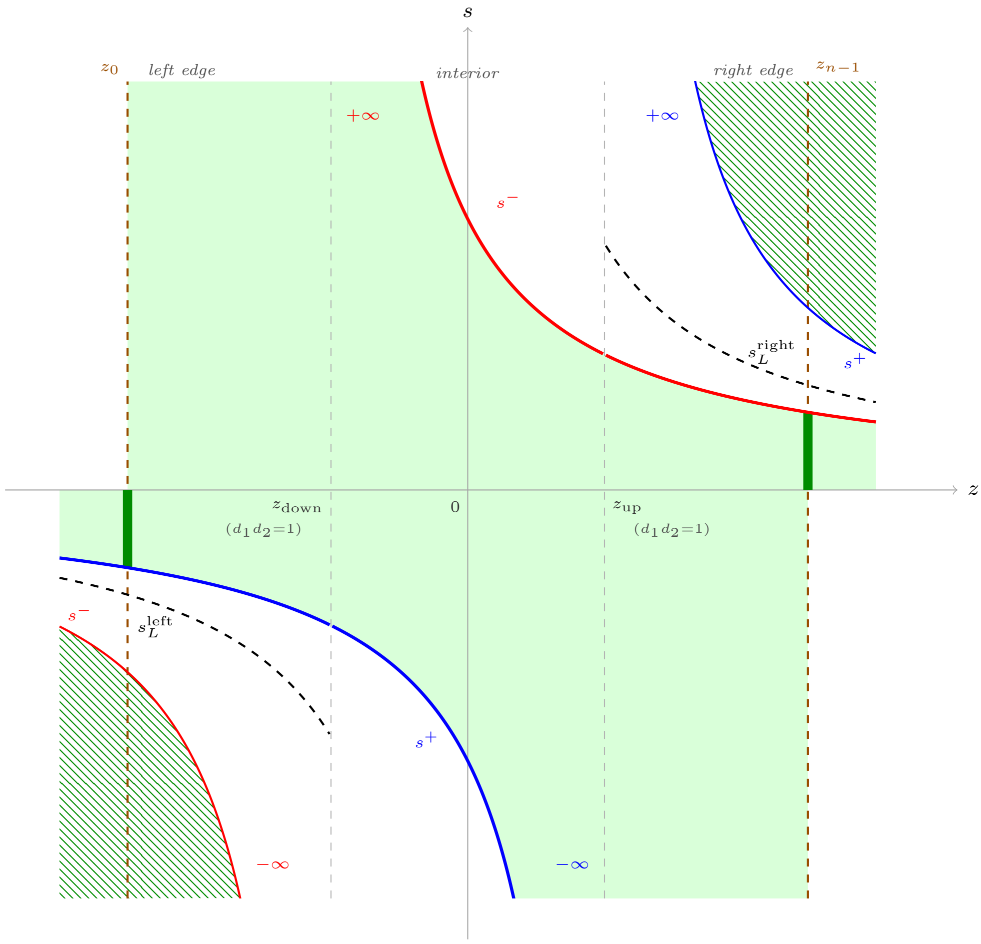

Figure 1: Root trajectories $s^{\pm}(z)$ at flat vol ($v = v_\star$, $T = 1$, $\sigma_\star = 20\%$). The two quadratic branches each have a single asymptote at one regime boundary and are smooth through the other; colors track the branches s^+ (blue) and s^- (red) uniformly, as fixed by (3). Bold (very thick) segments mark the lower-magnitude branch in each region (cap $s \le s^-$ on right wing and interior; floor $s \ge s^+$ on left wing and interior); thinner segments mark the larger-magnitude branch past the through-infinity escape (s^- past z_{down}; s^+ past z_{up}). The ordering reverses across each regime boundary because the denominator $d_1 d_2 - 1$ changes sign. The dashed black curves are the Lucic bounds s_L^{right} at the right edge and s_L^{left} at the left; the inequality (7) is visible as the strict interlacing $s^- < s_L^{\text{right}} < s^+$ at the right edge and $s^- < s_L^{\text{left}} < s^+$ at the left (where all three are negative). *Shading:* the union of solid and hatched green marks the butterfly-arbitrage-free region (PDF ≥ 0). Within it, the hatched strips lie above s_L^{right} at the right edge and below s_L^{left} at the left; these regions create vertical-spread arbitrage per [6] and are excluded by (7). Solid green is the retained admissible region. Orange dashed lines mark CVI's edge knots z_0, z_{n-1}; the caps at $s = 0$ in the tails are CVI-specific (linear wing extrapolation).

5 Infeasibility of the larger-magnitude root

The inequality (7) excludes s_+: any slice with $s = s_+$ at the right edge violates Lucic's slope bound. Let us look at the consequence. The CVI paper [1] selected the lower root on practical grounds (even the retained root already implies a skew at the limit of strike arbitrage; the upper root would be steeper still); here we show the selection follows from strike-arbitrage-freeness alone, not from empirical observation. There is no strike-arbitrage-free CVI smile with $s \geq s_+$ at the right edge (or $s \leq s_-$ at the left).

Two equivalent characterizations of strike-arbitrage-freeness. Following Carr and Madan [4] (Section 2, in continuous form), and assuming zero rates for simplicity, strike-arbitrage-freeness of an implied-volatility slice imposes two no-arbitrage tests on the call price function $C(K)$,

- *Convexity (butterfly):* $\partial^2 C/\partial K^2 \geq 0$, equivalently the risk-neutral density is non-negative.
- *Vertical spread:* $\partial C/\partial K \in [-1, 0]$, equivalently the risk-neutral CDF $\mathbb{P}(S_T \leq K) \in [0, 1]$, recovered via Breeden and Litzenberger from $\partial_K C = -(1 - \mathbb{P}(S_T \leq K))$.

together with the boundary conditions $C(0) = F$ at the small-strike end and $\lim_{K\to\infty} C(K) = 0$ at the large-strike end.

For a Black-Scholes-priced slice satisfying strict Lee (the setting throughout this note), this simplifies: the slice is strike-arbitrage-free iff the call price function $C(K)$ is convex in K; the remaining Carr-Madan conditions are automatic from the Black-Scholes formula. Roper [9] establishes the chain (Theorem 2.9 reduces strike-arbitrage-freeness to convexity plus $d_+ \to -\infty$; Proposition 2.11(1) supplies $d_+ \to -\infty$ from strict Lee); Gatheral and Jacquier [5] (Lemma 2.2) give the equivalent characterization (density-form plus large-strike limit).

Vertical-spread arbitrage at the upper root. With $s = s_+ > s_L^{\text{right}}$ at the right edge knot, Lucic's bound (6) is violated. As noted above, the bound is a necessary condition for absence of vertical-spread arbitrage [6], so the violation produces a vertical-spread arbitrage at the edge knot itself: a finite-strike call spread long $C(K_{n-1})$ and short $C(K_{n-1} + \epsilon)$ collects cash at inception with non-negative payoff at expiry. The mirror at the left edge is the put-side analogue.

Infeasibility of the larger-magnitude root. The two characterizations of strike-arbitrage-freeness must agree. Suppose a slice has $s \geq s_+$ at the right edge knot and is strike-arbitrage-free. Strike-arb-freeness gives global convexity of $C(K)$, strict Lee, and the Black-Scholes intrinsic-value lower bound (second characterization); the chain forces $\partial_K C \leq 0$ everywhere. But the previous paragraph established $\partial_K C > 0$ near K_{n-1} for any $s \geq s_+$, a contradiction. Hence:

$$\boxed{\text{No strike-arbitrage-free smile satisfies } s \geq s_+ \text{ at } z_{n-1} \text{ (or } s \leq s_- \text{ at } z_0).} \tag{9}$$

The mirror at the left edge follows by symmetry.

Why CVI's implementation does not separately enforce vertical-spread. The CVI paper [1] enforces butterfly globally via the interior optimization and controls the tail through the Lee admissibility criterion; under Black-Scholes pricing, the second characterization then gives strike-arbitrage-freeness directly, ruling out vertical-spread arbitrage without a separate constraint.

Why CVI's calibration excludes the upper root. The larger-magnitude-root exclusion (9) also explains why the CVI fit rejects this branch in practice: no strike-arb-free smile lands on s_+ at the edge, so the QP formulation solved by Clarabel returns infeasible if forced onto the upper root $s \geq s_+$ at the right edge. The CVI paper was right to exclude the root of larger magnitude but did not prove that such solutions do not exist.

Visualization. Figure 1 visualizes the situation in the 2D (z, s) slice at flat vol. At the right edge z_{n-1}, the values of s free of butterfly arbitrage form two disjoint regions (the solid and hatched green strips combined): the lower region $\{s \leq s^-\}$ and the upper region $\{s \geq s^+\}$. Lucic's bound selects the lower region as strike-arb-free; the upper region is free of butterfly arbitrage but has vertical-spread arbitrage, hence empty under strike-arb-freeness. The result (9) is the higher-dimensional version of the same picture: for any choice of parameters, the upper-root region $\{s \geq s_+\}$ in any strike-arb-free parametrization with linear-in-variance wings is empty. More generally, the exclusion applies at any point where $c = 0$ and $d_1 d_2 > 1$. §8 extends this to general c: the larger-magnitude root is excluded wherever it exists in the regime $d_1 d_2 > 1$.

6 Alternative formulation of the maximum-skew constraint

Having established the selection rule, let us rewrite (3) (the *singular* form, with $0/0$ at the regime boundary $d_1 d_2 = 1$) in an equivalent *regular* form.

The inequality (7) reduces the edge butterfly constraint to $s \leq s_-$ at the right edge; by the mirror, $s \geq s_+$ at the left. From (3), using the identity $(2z - \psi\sqrt{\Delta})(2z + \psi\sqrt{\Delta}) = 4\psi(d_1 d_2 - 1)$ that follows from Vieta's identity (4), each root admits a regular form:

$$s_- = \frac{4\psi}{2z + \psi\sqrt{\Delta}} \qquad \text{and} \qquad s_+ = \frac{4\psi}{2z - \psi\sqrt{\Delta}}.$$

Clearing the denominator at each edge, we obtain two regular constraints. At the right edge, the coefficient $2z + \psi\sqrt{\Delta}$ is positive (sum of two positives, since $z_{n-1} > 0$), so the inequality direction is preserved:

$$\boxed{\left(2z + \psi\sqrt{\Delta}\right) s \;\leq\; 4\psi \qquad \text{(right edge).}} \tag{10}$$

At the left edge, the coefficient $2z - \psi\sqrt{\Delta}$ is negative (sum of two negatives, since $z_0 < 0$), so dividing by it flips the inequality, which after rearrangement gives

$$\boxed{\left(2z - \psi\sqrt{\Delta}\right) s \;\leq\; 4\psi \qquad \text{(left edge).}} \tag{11}$$

Both regular forms are linear in s and equivalent to the constraints $s \leq s_-$ (right) and $s \geq s_+$ (left). Like the singular form, the regular form remains non-linear in v, which enters through ψ and Δ.

Numerical stability at the regime boundaries. In the singular form, the denominator $d_1 d_2 - 1$ of the retained root vanishes at the boundary. By Vieta's identity (4), so does the numerator: at $d_1 d_2 = 1$ we have $4z^2 = \psi^2\Delta$, so $(2z - \psi\sqrt{\Delta})(2z + \psi\sqrt{\Delta}) = 0$. The specific factor that vanishes is the one inside the retained root's numerator: at the right-edge regime boundary, we have $2z = \psi\sqrt{\Delta}$, so $2z - \psi\sqrt{\Delta} = 0$ (the numerator of s_-), and $s_- = 0/0$. At the left-edge regime boundary, we have $2z = -\psi\sqrt{\Delta}$, so $2z + \psi\sqrt{\Delta} = 0$ (the numerator of s_+), and $s_+ = 0/0$. The ratio cancels analytically, but numerical evaluation near the boundary amplifies cancellation error. In practice, for CVI, this is a non-issue: edge knots are placed well inside the regime $d_1 d_2 > 1$, not close to the boundary $d_1 d_2 = 1$.

In contrast, the regular forms (10) and (11) reduce at the respective regime boundary to $4zs \le 4\psi$ (using $\psi\sqrt{\Delta} = |2z|$) and are thus well behaved.

7 The maximum-skew constraint at the edge knot

When $d_1 d_2 > 1$, the surviving edge constraint after the exclusion of §5 is $s \le s_-$ at the right edge (with the mirror $s \ge s_+$ at the left). Under linear-in-variance wing extrapolation, this constraint admits a clean interpretation: it is the finite-strike form of Lee's asymptotic large-strike bound [3], strictly tighter than Lee at any finite knot and recovering Lee only in the limit. After listing the equivalent formulations of the constraint, we prove the strict tightening and show that edge enforcement propagates butterfly admissibility throughout the linear-extrapolated wing tail. Together, the maximum-skew constraint guarantees the finite-strike form of Lee's bound at the knot (large-strike at the right edge, small-strike at the left) and butterfly admissibility at and beyond the knot.

Equivalent formulations. The maximum-skew constraint admits two quotient expressions when $d_1 d_2 > 1$:

Singular at $d_1 d_2 = 1$ (0/0):	
Raw variables (CVI paper [1])	$s_\pm = \dfrac{2v\sqrt{T}\left(k \pm \sqrt{vT}\sqrt{1 + vT/4}\right)}{\sqrt{v_\star}\left(k^2 - v^2T^2/4 - vT\right)}$
Normalized variables (3)	$s_\pm = \dfrac{2z \pm \psi\sqrt{\Delta}}{d_1 d_2 - 1}$
Regular at $d_1 d_2 = 1$ (§6):	
Raw variables	$s_\pm = \dfrac{4v\sqrt{T}}{\sqrt{v_\star}\left(2k \mp \sqrt{vT(vT+4)}\right)}$
Normalized variables	$s_\pm = \dfrac{4\psi}{2z \mp \psi\sqrt{\Delta}}$

The constraint is $s \le s_-$ at the right edge and $s \ge s_+$ at the left. The two entries in the upper group are exactly the same expression in different notations; both have a 0/0 pathology at the regime boundary $d_1 d_2 = 1$. The lower entry, derived in §6 using the identity $(2z - \psi\sqrt{\Delta})(2z + \psi\sqrt{\Delta}) = 4\psi(d_1 d_2 - 1)$, is equivalent and regular at the boundary. The CVI paper uses only the singular raw form; the regular form is introduced in §6.

Tighter than Lee at any finite knot. Lee's asymptotic bound on the implied vol slope rewrites in CVI variables as $\sqrt{w_\star}\, s < 2$ at the right wing (§4, "Relation to Lee's bounds"). From the regular form in raw variables, $\sqrt{w_\star}\, s_- = 4w/(2k + \sqrt{w(w+4)})$. When $d_1 d_2 > 1$ (equivalently $4k^2 > w(w+4)$, i.e., $2k > \sqrt{w(w+4)}$), this condition tightens the denominator and gives the chain

$$\sqrt{w_\star}\, s_- \;=\; \frac{4w}{2k + \sqrt{w(w+4)}} \;<\; \frac{4w}{2\sqrt{w(w+4)}} \;=\; 2\sqrt{\frac{w}{w+4}} \;<\; 2,$$

strict at any finite edge knot. Since under linear-in-variance extrapolation past the edge knot s at the knot equals the asymptotic wing slope, $s \le s_-$ at the knot is Lee's bound enforced at the finite knot itself: strictly tighter than Lee's asymptotic statement. In the saturation limit $d_1 d_2 \to 1^+$ (regime-boundary, $2k \to \sqrt{w(w+4)}$), the first inequality becomes an equality, leaving $\sqrt{w_\star}\, s_-$ at the ceiling $2\sqrt{w/(w+4)}$, still strictly below 2 at any finite w. The second inequality closes only as $w \to \infty$ along the saturation curve $4k^2 = w(w+4)$, equivalently the edge knot pushed to infinity; this is quantified in the paragraph "Approaching Lee by placing the edge knot far out" below.

Tail propagation and the from-below approach. The CVI paper [1] stated, without explicit derivation, that the asymptotic limits of $s_\pm(k)$ derived in §4 are approached *from below* for s_- at the right wing (and *from above* for s_+ at the left) as $k \to \infty$. We derive this monotonicity here.

In this paragraph z runs along the linear tail past the right-edge knot at z_{n-1}, with $\psi(z) = \psi + s(z - z_{n-1})$ and $s = s_-$. The combination $sz - \psi(z) = sz_{n-1} - \psi$ is constant in z. Differentiating $s_-(z) = 4\psi(z)/(2z + \psi(z)\sqrt{\Delta(z)})$ along the tail and using $\psi^2\Delta = w_\star\psi^2 + 4\psi$ gives

$$\partial_z s_-(z) \;=\; \frac{8\big[\sqrt{\Delta(z)}\,(sz_{n-1} - \psi) + s\big]}{\sqrt{\Delta(z)}\,(2z + \psi(z)\sqrt{\Delta(z)})^2}. \tag{12}$$

The denominator is positive. For the numerator bracket $\sqrt{\Delta(z)}(sz_{n-1} - \psi) + s$, both z_{n-1} and $\psi = \psi(z_{n-1})$ are edge-knot values (constants); only $\sqrt{\Delta(z)}$ varies along the tail. So the bracket is a linear function of $\sqrt{\Delta(z)}$ with constant coefficient $(sz_{n-1} - \psi)$ and constant intercept s. If $sz_{n-1} \geq \psi$, the first term is non-negative and the bracket is positive throughout. If $sz_{n-1} < \psi$, the first term is negative; since $\sqrt{\Delta(z)} = \sqrt{w_\star + 4/\psi(z)}$ decreases as z grows along the tail (because $\psi(z)$ grows), the magnitude of this negative term shrinks with z, so the bracket is smallest at $z = z_{n-1}$ (where $\sqrt{\Delta(z)}$ is largest) and grows from there. Either way, monotonicity reduces to the inequality at the knot: $\sqrt{\Delta}(sz_{n-1} - \psi) + s > 0$. Substituting $s = 4\psi/(2z_{n-1} + \psi\sqrt{\Delta})$ from (3) and using $\psi^2\Delta = w_\star\psi^2 + 4\psi$ to simplify the numerator yields, after combining over the common denominator $2z_{n-1} + \psi\sqrt{\Delta} > 0$,

$$\sqrt{\Delta}(sz_{n-1} - \psi) + s \;=\; \frac{2\psi z_{n-1}\sqrt{\Delta} - w_\star\psi^2}{2z_{n-1} + \psi\sqrt{\Delta}}.$$

This is positive iff $2z_{n-1}\sqrt{\Delta} > w_\star\psi$, equivalently (squaring positive quantities) $4z_{n-1}^2\Delta > w_\star^2\psi^2$. Substituting $\Delta = w_\star + 4/\psi$ rewrites this as

$$4z_{n-1}^2 w_\star + 16z_{n-1}^2/\psi \;>\; w_\star^2\psi^2.$$

The regime condition $d_1d_2 > 1$ at the edge knot is $4z_{n-1}^2 > w_\star\psi^2 + 4\psi > w_\star\psi^2$; multiplying by $w_\star$ gives $4z_{n-1}^2 w_\star > w_\star^2\psi^2$, and adding the positive term $16z_{n-1}^2/\psi$ to the left preserves the inequality. Hence $\partial_z s_-(z) > 0$ throughout the linear tail, and $s_-(z) > s_-$ for every $z > z_{n-1}$, proving the *from-below* approach claimed in [1]; the mirror argument at the left edge gives the symmetric *from-above* approach for s_+.

As a corollary, the constant wing slope $s = s_-$ remains strictly inside the lower-root admissible region at every $z > z_{n-1}$, so butterfly admissibility at the edge knot propagates to the entire wing under linear extrapolation, with no separate enforcement needed.

Necessity and sufficiency at and beyond the knot.
Proposition 7.1 (CVI at-and-beyond equivalence)**.** *For a CVI smile (linear-in-variance wing extrapolation past the edge knot) satisfying strict Lee, under Black-Scholes pricing:*

$$s \leq s_-(z_{n-1}) \iff \textit{strike-arbitrage-free at and beyond } z_{n-1},$$

with mirror at the left edge:

$$s \geq s_+(z_0) \iff \textit{strike-arbitrage-free at and beyond } z_0.$$

Proof. Combining the strict Lee tightening with the tail propagation above, $s \le s_-$ at the right edge knot is necessary and sufficient for butterfly admissibility *and* Lee's tail bound throughout the linear-extrapolated wing. **Sufficiency for butterfly**: starting from $s \le s_-$ at the knot, tail-propagation monotonicity ($\partial_z s_-(z) > 0$) gives $s \le s_-(z)$ for all $z \ge z_{n-1}$. **Sufficiency for Lee**: the strict tightening gives $\sqrt{w_\star}\, s_- < 2$ at the knot; under CVI's linear-in-variance wing the skew s is constant past the knot, so the asymptotic Lee slope $\beta = s\sqrt{w_\star}$ is the same at the knot and at infinity, and $\sqrt{w_\star}\, s < 2$ holds throughout the wing. **Necessity**: by Carr-Madan (§5), strike-arbitrage-freeness at and beyond z_{n-1} implies butterfly admissibility, hence $s \le s_-(z)$ for all $z \ge z_{n-1}$; with $s_-(z)$ increasing along the wing, the knot is the tightest (binding) location, so this reduces to $s \le s_-$ at the knot. Black-Scholes pricing of the slice is implicit in the linear-in-variance wing setting (we work with the implied vol surface throughout); under it, butterfly admissibility plus Lee's tail bound throughout the wing imply strike-arbitrage-freeness throughout (§5). The mirror at the left edge follows by $k \to -k$ symmetry. ☐

Approaching Lee by placing the edge knot far out. Following the limit identified in the strict-tightening paragraph, for a reference $w_\star = 0.04$ ($\sigma_\star = 20\%$, $T = 1$), the table below tabulates the ceiling $\beta_{\max} = 2\sqrt{w/(w+4)}$ on $\sqrt{w_\star}\, s_-$ at various edge-knot positions z (with $w = w_\star\psi$ from $4z^2 = w_\star\psi^2 + 4\psi$), along with the corresponding minimum moment $p_{\min} = 1 + (2-\beta_{\max})^2/(8\beta_{\max})$ from Lee's moment formula [3] ($p_{\min}$ being the maximum order of finite moment of the underlying):

z at edge	$\beta_{\max}$	$p_{\min}$
2	0.39	1.85
3	0.55	1.47
5	0.83	1.21
10	1.24	1.06
20	1.56	1.02

For shorter-dated options, w on the saturation curve at a given z scales with $w_\star$, so matching a given $\beta_{\max}$ ceiling requires placing the edge knot at larger z.

8 Generalization to $c \neq 0$

So far we have analysed the maximum-skew constraint $s \le s_-$ at CVI's right edge knot, where linear-in-variance wing extrapolation forces $c = 0$ and the strike-arbitrage-freeness chain (§5) is closed via Black-Scholes pricing. In this section we relax the $c = 0$ assumption: we extend the larger-magnitude-root exclusion of §5 from $c = 0$ to general c, yielding a pointwise necessary-and-sufficient form of strike-arbitrage-freeness that combines the maximum-skew constraint $s \le s_-(z, c)$ with a call-spread Mills bound. Under Black-Scholes pricing the maximum-skew constraint alone is pointwise necessary and sufficient (Mills automatic via Roper's chain). We continue to call $s \le s_-$ at a right-wing point where $d_1 d_2 > 1$ (with mirror $s \ge s_+$ at a left-wing point) the *maximum-skew constraint*, the headline result of the paper (formalized as Theorem 9.2), with $s_\pm$ now also a function of the local convexity c. Extending the result from *pointwise* to *at and beyond* the boundary (the linear-extrapolation $c = 0$ statement of §7, generalized to a wing on which $c(z) \to 0$ smoothly) requires a tail-propagation condition on $c(z)$: §10 establishes this asymptotically (under $c(z) = o(1/z^2)$, covering SVI and SSVI); the general tail-propagation condition for arbitrary c is left as future work.

Notation introduced in this section.

- $s_-(z,c), s_+(z,c)$ (defined in the next paragraph): lower and upper butterfly roots at point z for local convexity c.
- $s_-^{\mathrm{CVI}} := s_-(z,0)$, $s_+^{\mathrm{CVI}} := s_+(z,0)$: the $c=0$ butterfly roots at any wing point z (specialized to the edge knot in §7).
- $\Delta(c) := \Delta^{\mathrm{CVI}} - 2c(d_1 d_2 - 1)/\psi$: convexity-adjusted discriminant; $\Delta^{\mathrm{CVI}} := \Delta(0) = w_\star + 4/\psi$ is the $c=0$ baseline (§2).
- $c^* := (w_\star \psi + 4)/(2(d_1 d_2 - 1))$: threshold for real butterfly roots ($\Delta(c^*) = 0$).

The butterfly quadratic at $c \neq 0$. The no-butterfly-arbitrage condition in CVI variables, in the general case where we do not assume $c = 0$, reads

$$\frac{v_\star (d_1 d_2 - 1)}{2v}\, s^2 + \sqrt{v_\star}\left(\frac{2d_1}{\sqrt{v}} - \sqrt{T}\right) s + 2 + c \;\geq\; 0, \tag{13}$$

which is (1) with the constant term shifted from 2 to $2+c$. The leading coefficient (unchanged) stays positive when $d_1 d_2 > 1$, and the two roots read

$$s_\pm(z,c) \;=\; \underbrace{\frac{2z \pm \psi\sqrt{\Delta(c)}}{d_1 d_2 - 1}}_{\text{singular}} \;=\; \underbrace{\frac{2\psi\,(2+c)}{2z \mp \psi\sqrt{\Delta(c)}}}_{\text{regular}},$$

where $\Delta(c)$ is the convexity-adjusted discriminant, in two equivalent forms

$$\Delta(c) \;=\; \Delta^{\mathrm{CVI}} - \frac{2c\,(d_1 d_2 - 1)}{\psi} \;=\; \left(1 + \frac{c}{2}\right)\Delta^{\mathrm{CVI}} - \frac{2cz^2}{\psi^2}.$$

Solving $\Delta(c^*) = 0$ in the first form gives the threshold

$$c^* \;=\; \frac{\psi\,\Delta^{\mathrm{CVI}}}{2(d_1 d_2 - 1)} \;=\; \frac{w_\star \psi + 4}{2(d_1 d_2 - 1)}.$$

Real roots exist iff $c \leq c^*$; past this threshold the butterfly constraint is non-binding regardless of the skew (no real roots means the upward parabola sits above zero for all s, so the inequality holds automatically). Qualitatively, increasing c raises s_- and lowers s_+, making the upper-side constraint $s \leq s_-$ (and the mirror $s \geq s_+$ at the left edge) easier to satisfy; the two roots merge at $c = c^*$.

$c < c^*$ in the deep tail. We can see that $c^*(z) > 0$ when $d_1 d_2 > 1$. We compare $c(z)$ against $c^*(z)$ asymptotically in three cases according to the growth of $\psi(z)$:

- **Superlinear** ($\psi(z)/z \to \infty$): ruled out by Lee's strict bound (§1), which forces $\psi(z)/z$ to stay bounded.
- **Linear** ($\psi(z)/z \to \beta/\sqrt{w_\star}$ with $0 < \beta < 2$, the standard case for implied-volatility parametrizations; the boundary $\beta = 0$ corresponds to sublinear ψ and is treated below). In this regime $\psi \sim \beta z/\sqrt{w_\star}$, so the two pieces of $d_1 d_2 = z^2/\psi - w_\star\psi/4$ satisfy $z^2/\psi \sim z\sqrt{w_\star}/\beta$ and $w_\star\psi/4 \sim \sqrt{w_\star}\,\beta z/4$; subtracting gives $d_1 d_2 - 1 \sim z\sqrt{w_\star}\,(1/\beta - \beta/4) = z\sqrt{w_\star}\,(4-\beta^2)/(4\beta)$, and $w_\star\psi + 4 \sim \sqrt{w_\star}\,\beta z$. Hence $c^*(z)$ converges to a positive constant:

  $$c^*(z) \;\longrightarrow\; \frac{2\beta^2}{4 - \beta^2} \;>\; 0$$

  (strictly positive since $0 < \beta < 2$). The asymptotically linear ψ naturally has $c(z) = \psi''(z) \to 0$ for smooth smiles (e.g., SVI/SSVI, with $c \sim z^{-3}$), so $c(z) < c^*(z)$ for z sufficiently large.

- **Sublinear (concave)** ($\psi(z)/z \to 0$ with ψ smoothly concave at infinity): the regime $d_1 d_2 > 1$ extends from §4 (derived there for linear-variance tails) to the sublinear case, since sublinearity gives $z/\psi \to \infty$, so z^2/ψ dominates $w_\star \psi/4$ in $d_1 d_2 = z^2/\psi - w_\star \psi/4$ and $d_1 d_2 \to \infty$. Concavity gives $c = \psi'' < 0$, while $c^* > 0$ from its formula, so $c(z) < 0 < c^*(z)$ for z large.

In both the linear and sublinear cases (i.e., the two cases not ruled out by Lee), the butterfly quadratic has two real roots in the deep tail, so the maximum-skew constraint $s \le s_-$ applies pointwise.

$s_-(z,c)$ vs Lee at finite strike. A side remark (used afterwards only in Figure 2): the maximum-skew *formula* $s_-(z,c)$ can exceed Lee's asymptotic ceiling $\sqrt{w_\star}\, s < 2$ at finite strike when c approaches c^* on a steep wing, even though the *achievable* slope remains Mills-constrained strictly below Lee. Working at a right-wing point (mirror at the left by $k \to -k$ symmetry), setting $c = c^*$ in $s_\pm$ gives $s_-(z, c^*) = 2z/(d_1 d_2 - 1)$ (the two roots merge). In the linear asymptotic case ($\psi/z \to \beta/\sqrt{w_\star}$), the §4 asymptotic $d_1 d_2 - 1 \sim z\sqrt{w_\star}\,(4-\beta^2)/(4\beta)$ gives

$$\sqrt{w_\star}\, s_-(z, c^*) \;=\; \frac{2z\sqrt{w_\star}}{d_1 d_2 - 1} \;\longrightarrow\; \frac{8\beta}{(2-\beta)(2+\beta)},$$

which exceeds 2 iff $\beta^2 + 4\beta - 4 > 0$, i.e., $\beta > 2(\sqrt{2} - 1) \approx 0.83$. So at a wing point on a smile with steep asymptotic slope ($\beta \gtrsim 0.83$), the maximum-skew formula $s_-(z,c)$ at high local convexity exceeds Lee's asymptotic ceiling; this is a property of the formula in isolation, not of the achievable slope, which Mills constrains strictly below Lee ((14) below).

Root / Mills ordering and larger-magnitude-root exclusion. We work at a right-wing point with $d_1 d_2 > 1$ and $c \le c^*$ (so the butterfly quadratic has real roots): $d_1 d_2 > 1$ holds asymptotically by strict Lee (§4, "Relation to Lee's bounds", for linear-variance tails; extended to sublinear ψ in the "$c < c^*$ in the deep tail" paragraph above), and $c \le c^*$ in the deep tail likewise by that paragraph. The §5 argument at $c = 0$ relies on the ordering $s_-^{\mathrm{CVI}} < s_{\mathrm{VS}}^{\mathrm{call}} < s_+^{\mathrm{CVI}}$, where $s_{\mathrm{VS}}^{\mathrm{call}} = 2\sqrt{\psi}\, R(f_2) > 0$ is the call-spread Mills threshold from §4, c-independent. We refer to the excluded branch (s_+ at the right, s_- at the left) as the *larger-magnitude root*, mirror-symmetric across the two wings; the rest of the paper uses this name.

Lemma 8.1 (Larger-magnitude-root exclusion)**.** *At any right-wing point ($d_1 d_2 > 1$, $z > 0$) with $c \le c^*$, no strike-arbitrage-free smile satisfies $s \ge s_+(z,c)$. (Mirror at the left wing: no strike-arbitrage-free smile satisfies $s \le s_-(z,c)$ at any left-wing point with $d_1 d_2 > 1$, $z < 0$, $c \le c^*$.)*

Proof. We show $s_+(z,c) > s_{\mathrm{VS}}^{\mathrm{call}}$ for all $c \le c^*$. s_+ decreases in c (since $\partial_c \Delta(c) = -2(d_1 d_2 - 1)/\psi < 0$ on the wing, and $s_+ = (2z + \psi\sqrt{\Delta(c)})/(d_1 d_2 - 1)$ with $z, \psi > 0$ and $d_1 d_2 - 1 > 0$) and attains its minimum over $c \le c^*$ at $c = c^*$, where $s_+|_{c=c^*} = 2z/(d_1 d_2 - 1)$ (the two roots merge); using Gordon's upper bound $R(f_2) < 1/f_2$ (§4) for $f_2 = z/\sqrt{\psi} + \sqrt{w_\star \psi}/2 > 0$,

$$s_{\mathrm{VS}}^{\mathrm{call}} \;=\; 2\sqrt{\psi}\, R(f_2) \;<\; \frac{2\sqrt{\psi}}{f_2} \;=\; \frac{2\psi}{z + \psi\sqrt{w_\star}/2}.$$

The condition $2z/(d_1 d_2 - 1) > 2\psi/(z + \psi\sqrt{w_\star}/2)$, with $d_1 d_2 - 1 = z^2/\psi - w_\star \psi/4 - 1$, reduces to $\psi(2z\sqrt{w_\star} + w_\star \psi + 4) > 0$, which holds for any $z > 0$, $\psi > 0$. Hence $s_+(z, c^*) > s_{\mathrm{VS}}^{\mathrm{call}}$, and by monotonicity $s_+(z,c) > s_{\mathrm{VS}}^{\mathrm{call}}$ throughout $c \le c^*$. Since call-spread Mills $s \le s_{\mathrm{VS}}^{\mathrm{call}}$ is necessary for strike-arbitrage-freeness (Lemma 3.7 of [6]), $s \ge s_+(z,c)$ would force $s > s_{\mathrm{VS}}^{\mathrm{call}}$, contradicting strike-arbitrage-freeness. The mirror at the left wing follows by $k \to -k$ symmetry, swapping call-spread Mills for put-spread Mills ($\dot\sigma \ge -R(-f_2)$, the put-side analogue of Lemma 3.7 of [6]). □

Lemma 8.1 generalizes the $c = 0$ exclusion of (9). Under Black-Scholes pricing (where Mills is automatic via Roper's chain, §5), the larger-magnitude-root exclusion gives that $s \le s_-(z, c)$ is pointwise necessary and sufficient for strike-arbitrage-freeness at any such point. This justifies the term *maximum-skew constraint* for $s \le s_-$: $s_-(z, c)$ is the maximum admissible skew at the point, and the larger-magnitude root is excluded wherever it exists in the regime $d_1 d_2 > 1$, i.e., when $c \le c^*$.

On the ordering of the lower root and Mills call spread. Let us further study how this ordering shifts with c, determining the binding side of the combined butterfly + Mills inequality (the non-BS pointwise form, eq. (15) below). With $s_+ > s_{\mathrm{VS}}^{\mathrm{call}}$ established, only the position of s_- relative to $s_{\mathrm{VS}}^{\mathrm{call}}$ varies, partitioning $c \le c^*$ into two ordering cases:

- **Butterfly-binding** ($s_- < s_{\mathrm{VS}}^{\mathrm{call}} < s_+$): the §5 ordering preserved.
- **Vertical-binding** ($s_{\mathrm{VS}}^{\mathrm{call}} \le s_-$): the lower-root threshold rises above the Mills threshold.

At $c = 0$, butterfly-binding holds (§4's ordering: $s_-^{\mathrm{CVI}} < s_{\mathrm{VS}}^{\mathrm{call}}$). Since s_- is strictly increasing in c, we split the analysis at $c = 0$:

- $c \le 0$: butterfly-binding holds, since $s_-(z, c) \le s_-^{\mathrm{CVI}} < s_{\mathrm{VS}}^{\mathrm{call}}$.
- $c \ge 0$: s_- rises with c from $s_-^{\mathrm{CVI}} < s_{\mathrm{VS}}^{\mathrm{call}}$ at $c = 0$ to $s_-(z, c^*) = s_+(z, c^*) > s_{\mathrm{VS}}^{\mathrm{call}}$ at c^* (by the proof of Lemma 8.1); by the intermediate-value theorem together with strict monotonicity, there is a unique crossing $c_{\mathrm{cross}} \in \,]0, c^*[$ with $s_-|_{c=c_{\mathrm{cross}}} = s_{\mathrm{VS}}^{\mathrm{call}}$; butterfly-binding holds on $[0, c_{\mathrm{cross}}[$ and vertical-binding on $]c_{\mathrm{cross}}, c^*[$.

For $c \ge c^*$, butterfly has no real roots and only Mills constrains ($s \le s_{\mathrm{VS}}^{\mathrm{call}}$).

On the ordering of the Mills call spread and Lee bound. We show that, under strict Lee, Mills's call-spread threshold sits strictly below Lee's asymptotic ceiling at every wing point in the regime $d_1 d_2 > 1$; a sharper asymptotic factorization further refines the comparison on a linear-in-variance wing. At any finite wing point in the regime $d_1 d_2 > 1$, $w(k) < 2k$ gives $\sqrt{w_\star \psi} < f_2$ (from $f_2 - \sqrt{w_\star \psi} = (2z - \sqrt{w_\star}\,\psi)/(2\sqrt{\psi})$, positive iff $4k^2 > w^2$, i.e., $w < 2k$); combined with Gordon's bound $R(f_2) < 1/f_2$, $\sqrt{w_\star}\, s_{\mathrm{VS}}^{\mathrm{call}} = 2\sqrt{w_\star \psi}\, R(f_2) < 2\sqrt{w_\star \psi}/f_2 < 2$ (Gordon for the first inequality, $\sqrt{w_\star \psi} < f_2$ for the second). The trailing < 2 is Lee's ceiling, so this establishes Mills $<$ Lee at every wing point in the regime. Furthermore, the rate at which Mills approaches Lee in the deep wing is captured by an asymptotic factorization. As the leading-order behaviour on the linear wing, $\psi(z) \sim \beta z/\sqrt{w_\star}$, so $\sqrt{w_\star}\,\psi \sim \beta z$, and $f_2 = z/\sqrt{\psi} + \sqrt{w_\star \psi}/2$ gives $\sqrt{w_\star \psi} \sim 2\beta f_2/(2 + \beta)$. Substituting into $s_{\mathrm{VS}}^{\mathrm{call}} = 2\sqrt{\psi}\, R(f_2)$:

$$\sqrt{w_\star}\, s_{\mathrm{VS}}^{\mathrm{call}} \;\sim\; \frac{4\beta}{2+\beta} \cdot f_2\, R(f_2). \tag{14}$$

With $f_2\, R(f_2) \to 1$ (Gordon-saturating) and $4\beta/(2 + \beta) < 2$ strict, the asymptotic limit is $\sqrt{w_\star}\, s_{\mathrm{VS}}^{\mathrm{call}} \to 4\beta/(2 + \beta) < 2$. Mills approaches Lee only in the simultaneous limit $\beta \to 2$ (saturating the second inequality) and $f_2 \to \infty$ (Gordon-saturating); either alone is insufficient. This parallels §7's "Approaching Lee" result for the lower root s_-^{CVI}, where, in the linear wing ($w \sim \beta k$), the formula $\sqrt{w_\star}\, s_- = 4w/(2k + \sqrt{w(w + 4)})$ of §7 factors asymptotically as $4\beta/(2 + \beta\eta)$ with $\eta \coloneqq \sqrt{1 + 4/w}$ (using $\sqrt{w(w + 4)} = w\eta$), giving the same simultaneous-limit saturation $\beta \to 2$ and $\eta \to 1$ (i.e., $w \to \infty$). Consequently, any smile satisfying vertical-spread no-arbitrage (automatic under Black-Scholes pricing with butterfly enforced, necessary under non-BS) has local slope strictly below Lee at every finite wing point, regardless of convexity c.

Pointwise strike-arbitrage-freeness without Black-Scholes. Without Black-Scholes pricing, Mills enters as an additional pointwise constraint alongside the maximum-skew constraint

$s \le s_-$. We record the combined form here for completeness. For the right-wing analysis we restrict to $s > 0$ (positive slope, the typical case for equity smiles), which sidesteps the put-spread Mills constraint $s \ge s_{\mathrm{VS}}^{\mathrm{put}}$ automatically: the put-spread threshold $s_{\mathrm{VS}}^{\mathrm{put}} := -2\sqrt{\psi}\,R(-f_2) < 0$ is negative, so $s > 0$ implies $s > s_{\mathrm{VS}}^{\mathrm{put}}$. At any right-wing point where $d_1 d_2 > 1$ and $s > 0$, two effective no-arbitrage conditions hold simultaneously: butterfly absence ($s \notin]s_-, s_+[$, i.e., the maximum-skew constraint $s \le s_-$ once the larger-magnitude root is excluded by Lemma 8.1) and call-spread Mills ($s \le s_{\mathrm{VS}}^{\mathrm{call}}$, necessary and sufficient for call-spread no-arbitrage; see §4). Combining these two conditions, the pointwise necessary-and-sufficient form on the right wing reads

$$\boxed{s \le \min\{s_-,\ s_{\mathrm{VS}}^{\mathrm{call}}\} \iff \text{strike-arbitrage-free at the point}\,.} \tag{15}$$

Given $d_1 d_2 > 1$ (assumed throughout this section), the inequality (15) pivots at c^*: for $c < c^*$ both butterfly and Mills are present (binding side switching at c_{cross}); for $c \ge c^*$ the butterfly is non-binding, s_- is undefined, and the min in (15) collapses to $s_{\mathrm{VS}}^{\mathrm{call}}$ alone. This is pointwise only.

Continuity at c^*: a discontinuity hidden by the Mills-ratio bound. The larger-magnitude-root exclusion would by itself create a discontinuity in the admissible set at c^*, which the Mills-ratio bound smoothly resolves. Indeed, without larger-magnitude-root exclusion, the butterfly forbidden interval $]s_-(z,c), s_+(z,c)[$ continuously shrinks to a single point at $c = c^*$ (the merged vertex) and vanishes for $c > c^*$, a continuous transition. However, with larger-magnitude-root exclusion the forbidden set is the half-line $]s_-(z,c), \infty[$ for $c < c^*$, shrinking to $]s_\pm(z,c^*), \infty[$ at $c = c^{*-}$ (still a non-empty half-line above the vertex), and would suddenly empty at $c > c^*$ where butterfly is non-binding, a discontinuity inconsistent with smooth c-dependence. Absence of vertical-spread arbitrage resolves this: Gordon's bound forces $c_{\mathrm{cross}} < c^*$ (§4's sandwich plus s_- monotone in c, Figure 2), the Mills threshold $s_{\mathrm{VS}}^{\mathrm{call}}$ takes over as the binding bound at c_{cross}, and the joint butterfly + Mills admissible set $\{s \le \min(s_-, s_{\mathrm{VS}}^{\mathrm{call}})\}$ equals $\{s \le s_{\mathrm{VS}}^{\mathrm{call}}\}$ on $]c_{\mathrm{cross}}, c^*[$, connecting continuously to the $c > c^*$ regime where butterfly is non-binding and Mills binds alone. The discontinuity created by the larger-magnitude-root exclusion is exactly compensated by the Mills-ratio bound taking over before c^*; Gordon's bound is what guarantees the compensation always works.

Mirror at the left edge. By $k \to -k$ symmetry, at any left-wing point with $d_1 d_2 > 1$ and $s < 0$, the active no-arbitrage failure is put-spread (with threshold $s_{\mathrm{VS}}^{\mathrm{put}} = -2\sqrt{\psi}\,R(-f_2) < 0$, the mirror to $s_{\mathrm{VS}}^{\mathrm{call}}$ at the right; the restriction $s < 0$ sidesteps the call-spread Mills, which is automatic since $s_{\mathrm{VS}}^{\mathrm{call}} > 0$). The right-edge results above mirror as: no strike-arbitrage-free smile satisfies $s \le s_-$ (the more-negative branch) when $c \le c^*$ (mirror of Lemma 8.1); and the pointwise necessary-and-sufficient form (mirror of (15)) reads

$$\boxed{s \ge \max\{s_+,\ s_{\mathrm{VS}}^{\mathrm{put}}\} \iff \text{strike-arbitrage-free at the point}\,.} \tag{16}$$

The CVI special case ($c = 0$ along with Black-Scholes pricing) is $s \ge s_+^{\mathrm{CVI}}$ at the left edge knot z_0, with the at-and-beyond extension established in Proposition 7.1 (mirror at the left).

Summary: $c = 0$ vs $c \ne 0$.

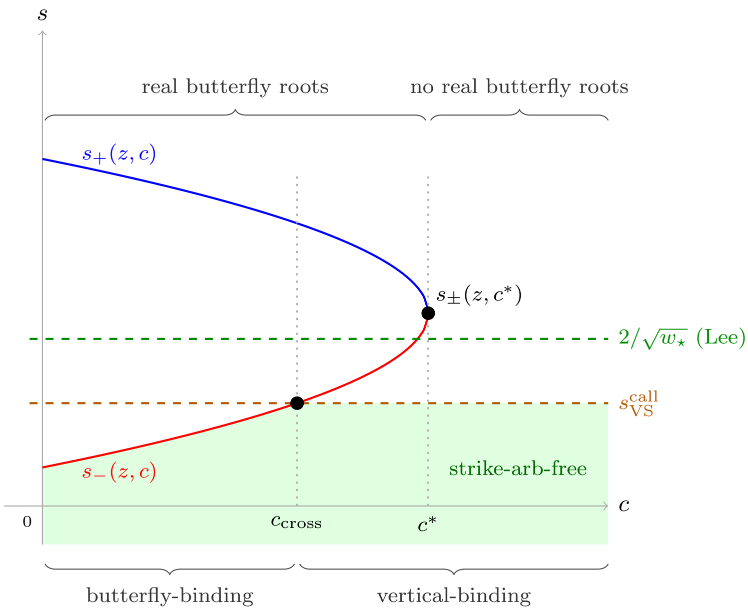

Figure 2: Butterfly roots $s_\pm(z,c)$ as functions of convexity c at a fixed right-wing point. The upper root $s_+(z,c)$ (blue) decreases in c; the lower root $s_-(z,c)$ (red) increases; they merge at $c = c^*$ where $\Delta(c^*) = 0$ (vertex value $s_\pm(z,c^*) = 2z/(d_1 d_2 - 1)$). The call-spread Mills threshold $s_{\mathrm{VS}}^{\mathrm{call}}$ (orange dashed) is independent of c and is sandwiched between the roots at $c = 0$ per §4; the lower root crosses Mills at the interior point c_{cross}. The green region $\{s \le \min(s_-(c), s_{\mathrm{VS}}^{\mathrm{call}})\}$ is the strike-arbitrage-free admissible set: bounded above by $s_-(c)$ on $[0, c_{\mathrm{cross}}]$ (butterfly-binding) and by $s_{\mathrm{VS}}^{\mathrm{call}}$ on $]c_{\mathrm{cross}}, \infty[$ (vertical-binding, with real butterfly roots for $c \le c^*$ and no real roots for $c > c^*$). Lee's asymptotic ceiling $\sqrt{w_\star}\, s < 2$ (green dashed) is positioned below the merge value here, depicting the steep-wing regime $\beta > 2(\sqrt{2} - 1) \approx 0.83$; for smaller β, Lee would sit above the merge. The Mills threshold (orange dashed) sits strictly below Lee at every finite wing point regardless of $\beta < 2$, by Gordon's bound combined with the regime $d_1 d_2 > 1$ (see §8; the asymptotic factorisation (14) gives the rate of approach in the deep wing). Gordon's bound guarantees the merge point sits above Mills ($s_{\mathrm{VS}}^{\mathrm{call}}$), so the upper root is always excluded by absence of vertical-spread arbitrage and the admissible region has a single connected component throughout $c \le c^*$, connecting continuously to the Mills-only region for $c > c^*$. The vertical-binding strip $s_{\mathrm{VS}}^{\mathrm{call}} < s \le s_-$ on $c \in\,]c_{\mathrm{cross}}, c^*[$ describes free local (s,c) pairs and is not reachable by a globally butterfly-free slice under Black-Scholes pricing and strict Lee (see the remark after Theorem 9.2).

	$c = 0$ (edge knot, linear wing)	$c \neq 0$ (general pointwise)
Discriminant	$\Delta = w_\star + 4/\psi$	$\Delta(c) = \Delta - 2c(d_1 d_2 - 1)/\psi$
Real roots	$d_1 d_2 > 1$	$d_1 d_2 > 1$ and $c \leq c^*$
Asymp. in wing (strict Lee)	regime $d_1 d_2 > 1$ holds (§4)	real roots exist (§8)
Max-skew constraint	$s \leq s_-$	$s \leq s_-$
Pointwise N&S under BS	$s \leq s_-$ alone (§5)	$s \leq s_-$ alone (§8)
Pointwise N&S without BS	$s \leq \min\{s_-,\ s_{\mathrm{VS}}^{\mathrm{call}}\}$	$s \leq \min\{s_-,\ s_{\mathrm{VS}}^{\mathrm{call}}\}$ (§8)
Tail propagation along wing	proven (§7)	asymp. (§10); full future work

At $c = 0$, pointwise N&S (necessary and sufficient) plus tail propagation along the linear wing reduces the constraint to a single-point check at the edge knot (§7). At $c \neq 0$, the pointwise N&S is established here. §10 derives two further asymptotic results: butterfly-binding along the wing under total-variation-finite skew with eventually monotone c, and single-strike propagation of butterfly admissibility under $c = o(1/z^2)$ (covering SVI / SSVI with $c \sim 1/z^3$). A full tail-propagation condition that would similarly reduce to a single-point check at arbitrary c is left as future work.

Practical sufficient condition. The pointwise necessary-and-sufficient form just established does not by itself extend to the entire wing past the chosen point. When $c \geq 0$ beyond z, s_-^{CVI} (the $c = 0$ value at z) is a practical sufficient pointwise bound, requiring no knowledge of the local convexity c: $s_-^{\mathrm{CVI}} \leq s_-(z, c)$ (since s_- is increasing in c), so $s \leq s_-^{\mathrm{CVI}}$ implies $s \leq s_-(z, c)$. Can this be extended to propagate butterfly admissibility along the entire wing past z? In two cases, yes. *First case*, CVI's exact linear-in-variance wing ($c \equiv 0$ past the edge knot, s constant): s_-^{CVI} propagates from the knot by §7 and coincides with the exact $s_-(z, c)$. *Second case*, SVI / SSVI (c decays as $O(1/z^3)$): §10 establishes that $s(z_b) \leq s_-(z_b, 0)$ at any wing strike past a parametrization-dependent threshold propagates butterfly admissibility to the entire wing past z_b (Proposition 10.1; explicit closed-form check for raw SVI, eq. (27)). The same propagation holds for any parametrization with asymptotic linear-in-variance wing and $c(z) = o(1/z^2)$. The slack between $s_-(z, c)$ and s_-^{CVI} remains the cost of the simpler $c = 0$ bound; on SVI / SSVI it is $O(c) = O(1/z^3)$, asymptotically tight.

9 Maximum-skew constraint theorem and root trajectories

Building on the $c \neq 0$ pointwise picture of §8, this section consolidates the pointwise picture across the smile (wings, interior, and degenerate no-real-roots cases) as a single parametrization-free theorem, then maps the geometric structure (root trajectories, regime boundary, label reconciliation across edges) that elaborates and visualizes it. We first record the interior characterization that the theorem will fold in.

Lemma 9.1 (Interior band feasibility). *At any interior point ($d_1 d_2 < 1$), the butterfly quadratic in s opens downward (leading coefficient $(d_1 d_2 - 1)/(2\psi) < 0$); butterfly absence is the closed band $s_+(z, c) \leq s \leq s_-(z, c)$ when the discriminant is non-negative ($c \geq c^*$, where $c^* < 0$ in the interior), and infeasible at the point when it is not ($c < c^*$, an unavoidable butterfly arbitrage).*

Proof. The butterfly quadratic in s at general c (eq. (13)) has leading coefficient $(d_1 d_2 - 1)/(2\psi)$, independent of c (the c-correction enters only via the constant term). For $d_1 d_2 < 1$, this is negative, so the parabola opens downward; the non-negativity region is the closed interval between the two real roots when they exist, and empty otherwise. The discriminant $\Delta(c)$ from §8 is non-negative iff $c \geq c^*$ in the interior (where c^* from the formula $c^* = (w_\star\psi + 4)/(2(d_1 d_2 - 1))$ is negative from the sign-flipped denominator). □

Theorem 9.2 (Maximum-Skew Constraint)**.** *For any implied-volatility parametrization satisfying strict Lee, at any point z of the smile with local skew $s = \partial_z\psi$ and convexity $c = \partial_z^2\psi$, under Black-Scholes pricing the slice is strike-arbitrage-free at the point if and only if, by regime:*

- *right wing ($d_1 d_2 > 1$, $z > 0$) with $c \leq c^*$: $s \leq s_-(z, c)$;*
- *interior ($d_1 d_2 < 1$) with $c \geq c^*$: $s_+(z, c) \leq s \leq s_-(z, c)$;*
- *left wing ($d_1 d_2 > 1$, $z < 0$) with $c \leq c^*$: $s \geq s_+(z, c)$;*
- *regime boundary ($d_1 d_2 = 1$): the quadratic degenerates to a linear condition with single finite root $(2 + c)\psi/(2z)$, giving the cap $s \leq (2 + c)\psi/(2z)$ at the right boundary z_{up} and the mirror floor $s \geq (2 + c)\psi/(2z)$ at the left boundary z_{down}.*

Outside the real-roots regime: on a wing ($c > c^*$), butterfly is non-binding and strike-arbitrage-freeness is automatic; in the interior ($c < c^*$), no choice of skew satisfies butterfly at the point, an unavoidable butterfly arbitrage.

The proof: under the slice-wide convention of §1, the local conditions below are the pointwise no-butterfly constraints to impose at each finite strike. When they are enforced globally across the slice, Black-Scholes pricing together with strict Lee implies slice-level strike-arbitrage-freeness (Roper's chain, §5), so no separate vertical-spread constraint is needed. **Right wing ($d_1 d_2 > 1$ strictly)**: real-roots sub-case ($c \leq c^*$) by larger-magnitude-root exclusion (Lemma 8.1) plus butterfly admissibility, giving the cap $s \leq s_-(z, c)$; no-real-roots sub-case ($c > c^*$) by the discriminant $\Delta(c)$ sign of §8 (with the leading coefficient $(d_1 d_2 - 1)/(2\psi) > 0$, no real roots $\Longrightarrow$ upward-opening parabola never crosses zero $\Longrightarrow$ butterfly automatic). **Regime boundary ($d_1 d_2 = 1$)**: $\Delta(c) = \Delta^{\mathrm{CVI}} > 0$ regardless of c (always in the real-roots regime), the larger-magnitude root escapes to infinity (merging mechanism below), and the cap $s \leq s_-$ continues via the surviving finite root $(2 + c)\psi/(2z)$, the regular form of §8. **Interior ($d_1 d_2 < 1$)**: Lemma 9.1 covers both the real-roots band ($c \geq c^*$, $s_+(z, c) \leq s \leq s_-(z, c)$) and the no-real-roots infeasibility ($c < c^*$). The left wing follows by $k \to -k$ symmetry. Without Black-Scholes pricing, the wing cap is replaced by the combined butterfly + Mills form $s \leq \min\{s_-,\ s_{\mathrm{VS}}^{\mathrm{call}}\}$ (right wing, (15)) and the mirror floor $s \geq \max\{s_+,\ s_{\mathrm{VS}}^{\mathrm{put}}\}$ (left wing, (16)).

Why the vertical-binding strip is unreachable. Figure 2's vertical-binding strip $s_{\mathrm{VS}}^{\mathrm{call}} < s \leq s_-$ on $c \in \,]c_{\mathrm{cross}}, c^*[$ depicts free local (s, c) pairs that satisfy the cap $s \leq s_-$ locally while violating call-spread Mills at the point. The slice-wide reading of *pointwise* introduced in §1 rules this configuration out: Roper's chain (§5) gives $P(K) \leq 1$ at every finite strike from global butterfly + strict Lee under Black-Scholes pricing, so the assumed local call-spread failure at K_0 is impossible. A globally butterfly-free slice therefore cannot occupy the vertical-binding strip, and the cap $s \leq s_-$ at the point is the binding constraint with Mills automatic. The same Roper-chain argument applies at any z and on either Mills side (call-spread upper, put-spread lower), covering Theorem 9.2's left-wing and interior-band cases by the same mechanism.

The regime boundary. The regime boundary is characterized by $d_1 d_2 = 1$, i.e. $z^2 = \psi(z) + w_\star\psi(z)^2/4$ (a self-consistency condition in z depending only on the local ψ, not on c; we denote any right-wing solution z_{up}, any left-wing solution z_{down}). Equivalently, in fixed-point form:

$$z_{\mathrm{up}} \;=\; \sqrt{\psi(z_{\mathrm{up}}) + w_\star\psi(z_{\mathrm{up}})^2/4}, \qquad z_{\mathrm{down}} \;=\; -\sqrt{\psi(z_{\mathrm{down}}) + w_\star\psi(z_{\mathrm{down}})^2/4}.$$

For the flat-vol example of Figure 4 ($\psi \equiv 1$), $z_{\mathrm{up}} = -z_{\mathrm{down}} = \sqrt{1 + w_\star/4} \approx 1.005$.

As z moves inward from either edge, d_1d_2 drops past 1 at the respective boundary. Below this threshold the leading coefficient $(d_1d_2 - 1)/(2\psi)$ of the butterfly quadratic (using $v = v_\star\psi$) flips sign: the parabola opens downward, and the feasibility region switches from "outside $[s_-, s_+]$" (two disjoint branches) to "inside $[s_+, s_-]$" (one connected interval around the money).

The merging mechanism at the boundary. At either boundary exactly, the quadratic term vanishes and the butterfly condition degenerates to a linear inequality. Let us see what this does to the two roots. From the edge side approaching the boundary, one root escapes to infinity while the other stays finite:

- On the right ($z \to z_{\rm up}^+$), $s_+ \to +\infty$ and s_- is the surviving finite root.
- On the left ($z \to z_{\rm down}^-$), by $k \to -k$ symmetry, $s_- \to -\infty$ and s_+ is finite.

Crossing into the interior regime, the ordering reverses to $s_+ < s_-$. The denominator $d_1d_2 - 1$ passes through zero at each boundary, sending the escaped branch through infinity; in the (s, z)-plane the feasibility boundary is a hyperbola-like curve with vertical asymptotes at $z = z_{\rm up}$ and $z = z_{\rm down}$.

Reconciling selection rules across edges. This mechanism reconciles what would otherwise look like a paradox. At each edge the ordering is $s_- < s_+$, yet the smile satisfies $s \le s_-$ at the right edge (below both roots, both positive) and $s \ge s_+$ at the left edge (above both roots, both negative). Same root ordering at both edges, opposite placements relative to it. In the interior, the ordering reverses: $s_+ < s_-$, and feasibility is the interval $s_+ \le s \le s_-$ (Lemma 9.1). The apparent inconsistency ($s_+ > s_-$ at the edges, $s_+ < s_-$ in the interior) is resolved by the regime flip: at each boundary the denominator $d_1d_2 - 1$ passes through zero, sending one root through infinity. The labels $s_\pm$ are fixed by (3); it is their ordering that reverses, not the labels themselves.

Geometric reading: where each bound lives. Globally, the cap $s \le s_-$ and the floor $s \ge s_+$ each live on the connected piece of the (z, s)-plane where their root is finite, and each dies at the singular value of z where its root escapes through infinity (past the singularity it is no longer defined). The cap dies at $z_{\rm down}$ (where $s_- \to -\infty$) and is alive to its right; the floor dies at $z_{\rm up}$ (where $s_+ \to +\infty$) and is alive to its left. On the right wing only the cap survives, the maximum-skew constraint of §4; on the left wing only the floor, the mirror; in the interior both are alive, and their conjunction $s_+ \le s \le s_-$ is the band geometry of the present section. Under Black-Scholes pricing (Mills automatic via Roper's chain, §5), this gives the pointwise equivalence (the abstract's cap-on-skew framing made literal: a one-sided cap on each wing, a two-sided band in the interior)

$$\text{strike-arbitrage-free at the point} \underset{\text{real roots}}{\iff} \begin{cases} s \le s_- & d_1d_2 \ge 1,\ z > 0 \text{ (right wing)} \\ s_+ \le s \le s_- & d_1d_2 < 1 \text{ (interior)} \\ s \ge s_+ & d_1d_2 \ge 1,\ z < 0 \text{ (left wing)} \end{cases} \tag{17}$$

(real roots: $c \le c^*$ on the wings, $c \ge c^*$ in the interior). When these conditions on c fail (no real roots: $c > c^*$ on the wings, $c < c^*$ in the interior),

$$\text{strike-arbitrage-free at the point} \underset{\text{no real roots}}{\iff} \begin{cases} \text{any } s \text{ (butterfly non-binding)} & d_1d_2 > 1 \text{ (wing)} \\ \text{no } s \text{ (butterfly infeasible)} & d_1d_2 < 1 \text{ (interior)} \end{cases} \tag{18}$$

At the regime boundary itself ($d_1d_2 = 1$), the c-correction in the discriminant ($\Delta(c) = \Delta^{\rm CVI} - 2c(d_1d_2 - 1)/\psi$, see §8) vanishes, so $\Delta(c) = \Delta^{\rm CVI} > 0$ regardless of c and the boundary always lies in the real-roots regime (eq. (17)). The wing cases of (17) continue to apply via the regular form of §6: the surviving root is $(2+c)\psi/(2z)$ in each (s_- at the right boundary, s_+ at the left; ψ/z at $c=0$); the other root goes through infinity.

Real butterfly roots require $\Delta(c) \geq 0$ (recalled from §8), equivalent to $c \leq c^*$ on the wings and $c \geq c^*$ in the interior (see Lemma 9.1's proof for the c^* formula and sign). Past the respective threshold the parabola has no zero crossings: on the wings (opening upward) butterfly is non-binding regardless of s; in the interior (opening downward) satisfying butterfly is infeasible at the point. Figure 5 maps these regions in the (z, c)-plane. For typical $c \geq 0$ the interior threshold is automatic ($c^* < 0 \leq c$), so the no-real-roots case bites in the interior only on sufficiently concave smiles. Under Black-Scholes pricing the wing's non-binding case is automatically strike-arbitrage-free at the point (§5's Roper chain); without Black-Scholes the Mills thresholds $s_{\text{VS}}^{\text{call}}$ (cap) and $s_{\text{VS}}^{\text{put}}$ (floor) continue to bind in the wings' no-real-roots regime. In the wing (where larger-magnitude-root exclusion is in force), the apparent discontinuity in the admissible set as c crosses c^* (butterfly cap binding below, suddenly non-binding above) is resolved smoothly by the call-spread Mills bound taking over before c^*; see §8's "Continuity at c^*" paragraph.

Figure 3 illustrates the three regimes at a glance, and Figure 4 plots the roots $s_\pm(z)$ as functions of z, making the ordering reversal through infinity explicit.

Approach to the singularity and re-emergence. Tracking $s_+(z)$ along the right half of the smile: in the interior, s_+ is the lower band-edge root; as z moves toward the right regime boundary z_{up}, s_+ descends and asymptotes to $-\infty$ at z_{up}, crossing the vertical-spread put bound $s_{\text{VS}}^{\text{put}}$ on the way down. Past z_{up}, s_+ re-emerges from $+\infty$ on the wing side as the larger-magnitude positive root, above the vertical-spread call bound $s_{\text{VS}}^{\text{call}}$. The mirror mechanism applies at the left boundary z_{down} via s_-. The singularity at $d_1 d_2 = 1$ is plainly present in the equations but unreachable from any strike-arbitrage-free smile: $s_{\text{VS}}^{\text{put}}$ and $s_{\text{VS}}^{\text{call}}$ stay finite on either side of the asymptote.

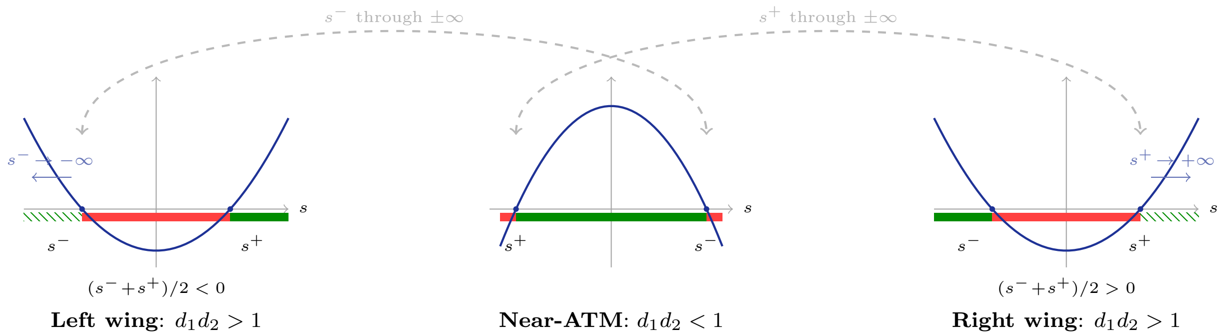

Figure 3: The butterfly quadratic in s across three regimes. In the wings ($z > z_{\text{up}}$ or $z < z_{\text{down}}$, $d_1 d_2 > 1$) the parabola opens upward and feasibility is *outside* the forbidden interval $[s^-, s^+]$, giving two branches; near-the-money ($z_{\text{down}} < z < z_{\text{up}}$, $d_1 d_2 < 1$) the leading coefficient flips sign, the parabola opens downward, and feasibility is the *inside* interval $[s^+, s^-]$. *Shading:* green (solid and hatched combined) marks the butterfly-arbitrage-free region; the hatched strip within it creates vertical-spread arbitrage per [6] and is excluded by (7); solid green is the retained branch. Red marks the butterfly-arbitrage violation. Arrows on the edge panels mark the root that escapes to $\pm\infty$ as z approaches the regime boundary; the dashed cross-panel arcs link the same labeled root re-emerging in the adjacent regime after the through-infinity jump.

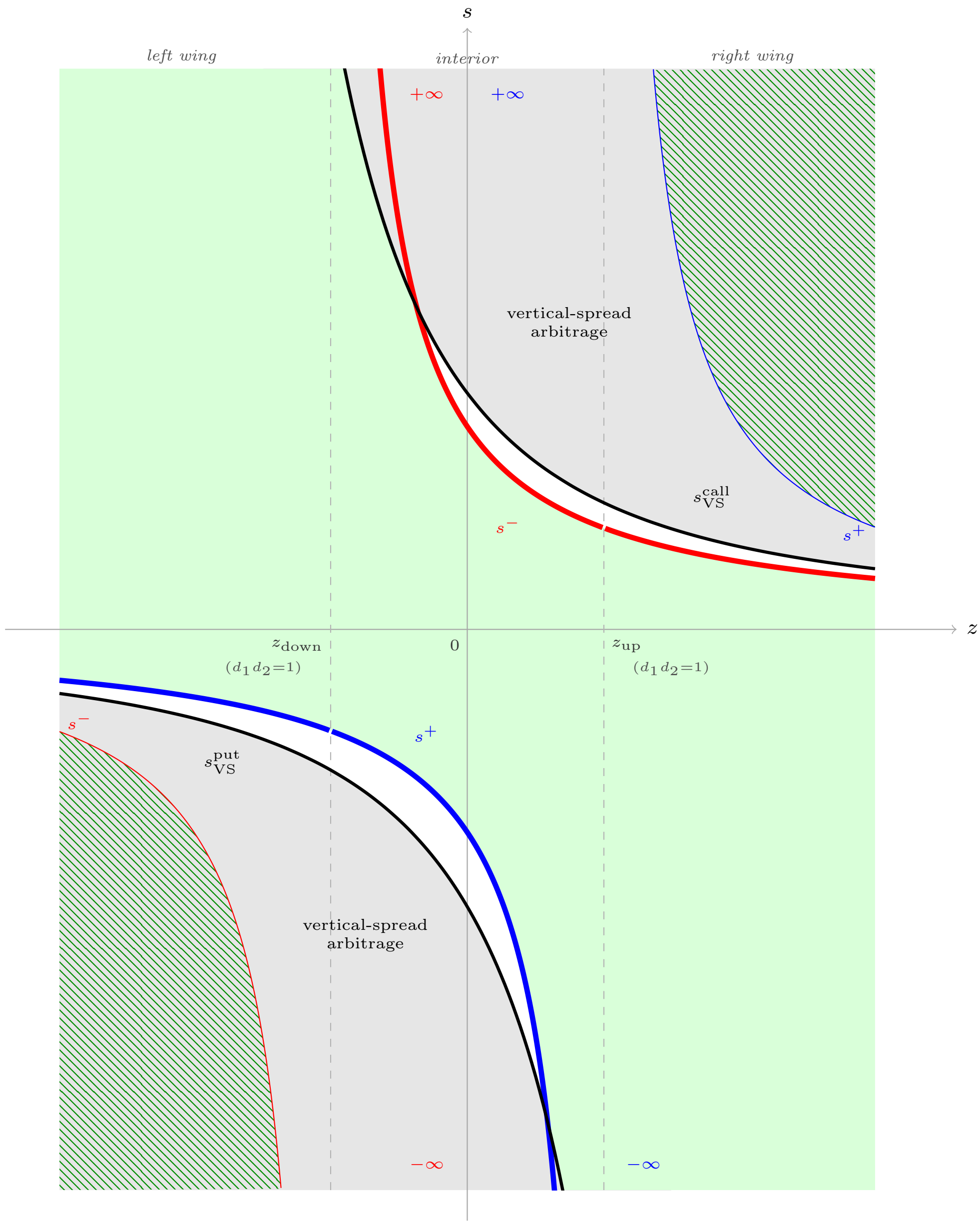

Figure 4: Root trajectories $s^{\pm}(z)$ across the smile at flat vol ($v = v_\star$, $T = 1$, $\sigma_\star = 20\%$), with the call-spread and put-spread Mills thresholds overlaid. Each butterfly branch escapes through infinity at one regime boundary ($d_1 d_2 = 1$) and re-emerges with opposite sign past it; bold segments mark the lower-magnitude branch, thinner segments the larger-magnitude branch. Solid black: $s^{\text{call/put}}_{\text{VS}} = \pm 2\sqrt{\psi}\, R(\pm f_2)$ on the right and left wings, with $f_2 = z/\sqrt{\psi} + \sqrt{w_\star \psi}/2$; the $\sqrt{w_\star}/2$ Itô drift in f_2 breaks $k \to -k$ symmetry, mildly visible here. *Shading:* green marks the butterfly-arbitrage-free region (PDF ≥ 0). The hatched green strips lie above $s^{\text{call}}_{\text{VS}}$ on the right wing and below $s^{\text{put}}_{\text{VS}}$ on the left, the butterfly-feasible larger-magnitude root excluded by vertical-spread arbitrage. Grey: vertical-spread arbitrage.

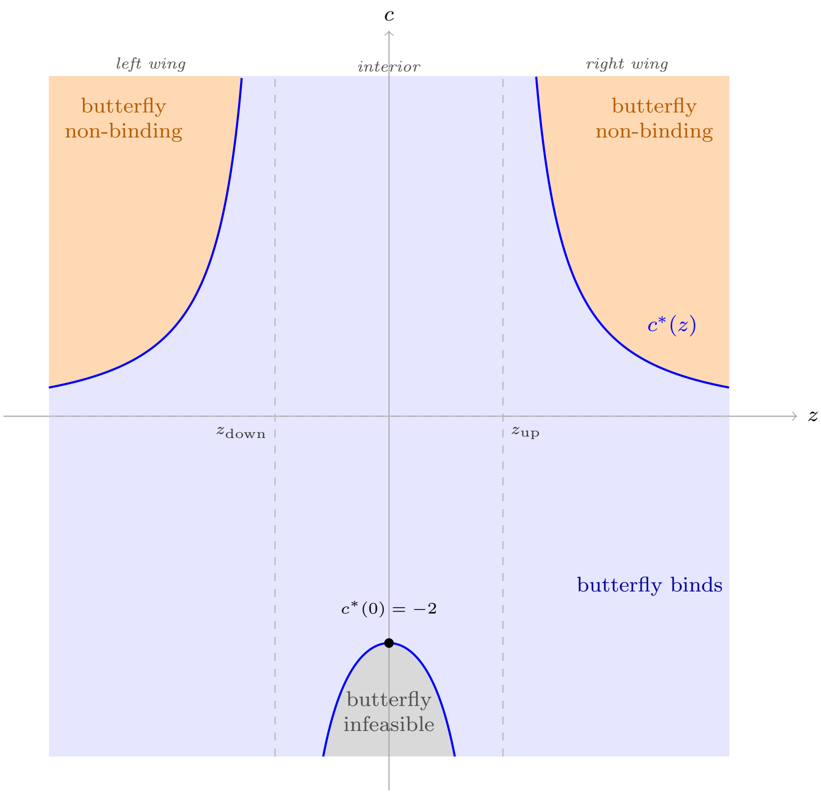

Figure 5: The convexity threshold $c^*(z) = (w_\star\psi + 4)/(2(d_1d_2 - 1))$ at flat vol ($v = v_\star$, $T = 1$, $\sigma_\star = 20\%$), partitioning the (z, c)-plane into binding and non-binding-or-infeasible regions. The blue curve has vertical asymptotes at the regime boundaries $z_{\text{up}}, z_{\text{down}}$ (where $d_1d_2 = 1$) and flips sign across each. On the wings ($d_1d_2 > 1$), c^* is positive and real butterfly roots require $c \leq c^*$; past the threshold the upward parabola has no zero crossings and butterfly is non-binding (orange). In the interior ($d_1d_2 < 1$), the same formula gives $c^* < 0$ from the sign-flipped denominator, real roots require $c \geq c^*$ instead, and past the threshold the downward parabola sits below zero and satisfying butterfly is infeasible at the point (grey), an unavoidable butterfly arbitrage regardless of skew. The ATM value $c^*(0) = -2$ at flat vol matches the classical convexity floor (at $s = 0$ butterfly reduces to $c \geq -2$). The dashed $c = 0$ horizontal sits inside the real-roots region across all regimes, approaching the boundary of that region only asymptotically deep in the wing (where $c^*(z) \to 0^+$); for typical convex smiles ($c \geq 0$) the no-real-roots case bites only on the wings. Linear-wing parametrizations ($\beta > 0$) lift the wing asymptote to a positive constant $2\beta^2/(4 - \beta^2)$ (derived in §8, "$c < c^*$ in the deep tail" paragraph) in place of the degenerate $c = 0$ asymptote shown here for $\beta = 0$.

10 Wing-asymptotic decay of c_{cross} and SVI / SSVI single-strike check

In §8 we saw that at every wing point a threshold $c_{\text{cross}}(z)$ splits the butterfly-real-roots range $c \le c^*$ into butterfly-binding ($c \le c_{\text{cross}}$) and vertical-binding ($c_{\text{cross}} < c \le c^*$); past c^*, butterfly roots disappear and only Mills constrains. Here we start by studying how $c_{\text{cross}}(z)$ behaves as we move out along the wing, in the asymptotic regime $\psi(z) \sim (\beta/\sqrt{w_\star})\, z$ as $z \to \infty$ (with $\beta \in]0, 2[$ strict Lee, and $k := z\sqrt{w_\star}$ the log-moneyness). Throughout this section "linear wing" refers to this asymptotic regime, which encompasses both CVI's exact linear-in-variance extrapolation past the edge knot (where ψ is literally affine in z and $c \equiv 0$) and parametrizations only asymptotically linear, $\psi(z) = (\beta/\sqrt{w_\star})\, z + O(1)$, including SVI and SSVI where $c(z) = O(1/z^3)$.

What happens is that the partition compresses as z grows: $c_{\text{cross}}(z)$ tends to zero at rate $1/z$, while staying strictly positive at every finite wing point. For smiles with regular wing decay, a total-variation argument on the skew gives $c(z) = o(1/z)$, so c decays strictly faster than c_{cross}, and butterfly-binding holds asymptotically with vanishing-but-positive margin.

As a second result, we extend the at-and-beyond propagation of §7 (CVI's exact linear wing) asymptotically: for SVI and SSVI ($c(z) = O(1/z^3)$), the check $s(z_b) \le s_-(z_b, 0)$ at any wing strike past a parametrization-dependent threshold propagates butterfly admissibility along the entire wing past z_b, with the closed form worked out for raw SVI.

Asymptotic expansion of $s_-(z,0)$ and $s^{\text{call}}_{\text{VS}}$. We expand both quantities asymptotically: $s_-(z, 0)$ is the lower butterfly root at $c = 0$, and $s^{\text{call}}_{\text{VS}}(z) = 2\sqrt{\psi}\, R(f_2)$ is the Mills threshold, which is independent of c. Both share the same leading-order limit $\sqrt{w_\star}\, s \to 4\beta/(2+\beta)$ (Lucic's saturating slope, §4); the expansions below give their $O(1/k)$ rates of approach to this common limit. The eq. (14) factorization $\sqrt{w_\star}\, s^{\text{call}}_{\text{VS}} \sim (4\beta/(2+\beta))\, f_2\, R(f_2)$, combined with Gordon's expansion $f_2\, R(f_2) = 1 - 1/f_2^2 + O(f_2^{-4})$ and the linear-wing identity $f_2^2 = (2+\beta)^2 k/(4\beta)$ (from $f_2 = (k + w/2)/\sqrt{w}$ with $w = \beta k$), gives

$$\sqrt{w_\star}\, s^{\text{call}}_{\text{VS}}(z) \;=\; \frac{4\beta}{2+\beta} - \frac{16\beta^2}{(2+\beta)^3\, k} + O(k^{-2}). \tag{19}$$

The parallel factorization $\sqrt{w_\star}\, s_-(z, 0) = 4\beta/(2 + \beta\eta)$ with $\eta := \sqrt{1 + 4/w}$ (§7 "Approaching Lee") expands via $\eta = 1 + 2/(\beta k) + O(k^{-2})$ to

$$\sqrt{w_\star}\, s_-(z, 0) \;=\; \frac{4\beta}{2+\beta} - \frac{8\beta}{(2+\beta)^2\, k} + O(k^{-2}). \tag{20}$$

Both quantities approach the same Lee saturation limit $4\beta/((2 + \beta)\sqrt{w_\star})$ from below, with negative $O(1/k)$ corrections whose difference is the gap quantified next (Figure 6 below). The expansions (19) and (20) hold for the pure-asymptotic linear wing $w(k) = \beta k$ used in the derivation; on a wing with non-zero intercept w_0 (e.g., CVI past the edge knot, SVI/SSVI under their natural parametrizations) an identical $+8w_0/((2 + \beta)^2 k)$ correction enters both expansions and cancels in the gap below, as worked out in the "Independence from the wing intercept" remark.

Asymptotic gap. Subtracting (20) from (19):

$$\begin{aligned}
\sqrt{w_\star}\big(s^{\text{call}}_{\text{VS}}(z) - s_-(z, 0)\big) \;&=\; \frac{8\beta}{(2+\beta)^2\, k} - \frac{16\beta^2}{(2+\beta)^3\, k} + O(k^{-2}) \\
&=\; \frac{8\beta}{(2+\beta)^3\, k}\big[(2+\beta) - 2\beta\big] + O(k^{-2}) \\
&=\; \frac{8\beta(2-\beta)}{(2+\beta)^3\, k} + O(k^{-2}).
\end{aligned} \tag{21}$$

Under strict Lee $\beta \in]0,2[$ the leading term is strictly positive, giving the deep-wing $O(1/k)$ rate at which the §4 sandwich gap $s_{\mathrm{VS}}^{\mathrm{call}} - s_-(z,0) > 0$ (known positive at every wing point with $d_1 d_2 > 1$) shrinks.

Independence from the wing intercept. The $O(1/k)$ coefficients displayed in (19) and (20) are the values for the pure-asymptotic case $w(k) = \beta k$ (no intercept), used in the derivation just above. Both expansions share the same leading Lee-saturation level $s_0(\beta) = 4\beta/(2+\beta)$, evaluated at the effective wing slope $\beta_{\mathrm{eff}} := w(k)/k$. A wing perturbation $\delta w(k) = w(k) - \beta k$ shifts β_{eff} by $\delta\beta_{\mathrm{eff}} = \delta w/k$, and since both Mills and butterfly inherit the leading s_0-dependence on β_{eff}, both receive an identical leading correction $\delta s = (\partial s_0/\partial\beta)\,\delta\beta_{\mathrm{eff}} = 8\delta w/((2+\beta)^2 k)$. Concretely, on a wing with non-zero intercept $w_0 := w_b - \beta k_b$ (the case on CVI's exact wing past the edge knot, and on SVI/SSVI under their natural parametrizations), redoing the expansions with $w(k) = \beta k + w_0$ adds an identical correction $+8w_0/((2+\beta)^2 k)$ to both (19) and (20); the corrections cancel exactly in the subtraction, so (21) and consequently (23) below are independent of w_0 at leading order. The same intercept-cancellation mechanism extends to higher-order wing corrections: for asymptotically linear (rather than strictly linear) wings such as SVI's right-wing expansion $w(k) = \beta k + w_0 + b\sigma^2/(2k) + O(k^{-3})$, the $b\sigma^2/(2k)$ correction enters Mills and butterfly identically at $O(1/k^2)$ in $\sqrt{w_\star}\,s$ and again cancels in the gap; hence (21) and (23) hold for any wing of the form $w(k) = \beta k + w_0 + o(1)$, covering both strictly linear (CVI past the edge knot, $o(1) \equiv 0$) and asymptotically linear (SVI, SSVI) cases.

While the gap and c_{cross} asymptotics are intercept-independent at leading order, the direction in which Mills and butterfly approach the common Lee saturation level $4\beta/(2+\beta)$ does depend on w_0 (via the sign of the $1/k$ correction in each of (19) and (20) after the intercept shift): both approach from below when $w_0 < 2\beta^2/(2+\beta)$ (the case depicted in Figure 6 at $w_0 = 0$), Mills from above and butterfly from below when $2\beta^2/(2+\beta) < w_0 < \beta$, and both from above when $w_0 > \beta$.

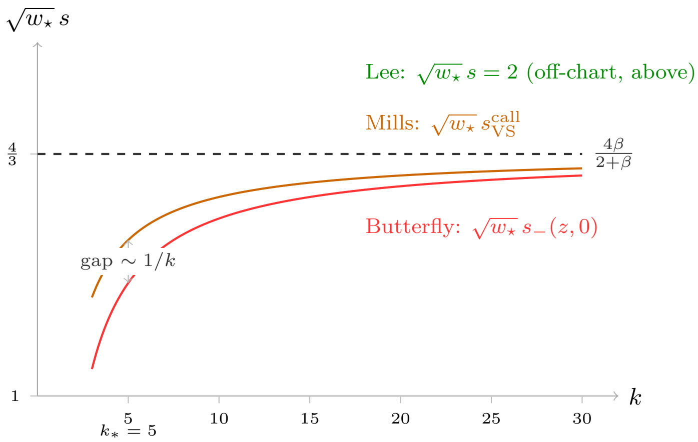

Figure 6: Asymptotic convergence of the Mills and butterfly thresholds to the common Lee saturation level $4\beta/(2+\beta)$ on the linear wing, depicted at $\beta = 1$, $w_0 = 0$ (pure linear wing, exact for CVI past the edge knot). Both $\sqrt{w_\star}\,s_{\mathrm{VS}}^{\mathrm{call}}$ (orange) and $\sqrt{w_\star}\,s_-(z,0)$ (red) approach the common limit (gray dashed) from below in this case, with butterfly sitting below Mills at every finite wing point; the vertical gap $\sim 8\beta(2-\beta)/((2+\beta)^3\,k)$ shrinks as $1/k$ (eq. (21)) and generates the butterfly-binding range $c_{\mathrm{cross}}(z) \sim 1/k$ (eq. (23)). Lee's ceiling at $\sqrt{w_\star}\,s = 2$ (green dashed) sits strictly above the common limit $4\beta/(2+\beta) < 2$ for any $\beta < 2$ (strict Lee). The non-zero intercept ($w_0 \neq 0$) case (regime-dependent direction of approach) is discussed in the "Independence from the wing intercept" remark after (21).

Asymptotic of $c_{\text{cross}}(z)$. The transition point $c_{\text{cross}}(z)$ is defined implicitly by $s_-(z, c_{\text{cross}}) = s_{\text{VS}}^{\text{call}}(z)$. Linearising $s_-(z,c) \approx s_-(z,0) + c\,\partial_c s_-(z,0)$ for small c,

$$c_{\text{cross}}(z) \;\approx\; \frac{s_{\text{VS}}^{\text{call}}(z) - s_-(z,0)}{\partial_c s_-(z,0)}. \tag{22}$$

With $\partial_c s_-(z,0) = 1/\sqrt{\Delta^{\text{CVI}}} \to 1/\sqrt{w_\star}$ asymptotically (since $\Delta^{\text{CVI}} = w_\star + 4/\psi \to w_\star$) and the gap (21),

$$\boxed{c_{\text{cross}}(z) \;\sim\; \frac{8\beta(2-\beta)}{(2+\beta)^3\, k} \qquad (k = z\sqrt{w_\star} \to \infty).} \tag{23}$$

The butterfly-binding admissible range $[0, c_{\text{cross}}(z)]$ shrinks to zero at rate $1/z$ but remains strictly positive at every finite wing point.

Asymptotic butterfly-binding under eventually monotone convexity. Assume on the wing: (i) c eventually monotone non-increasing; (ii) the linear-wing hypothesis $s(z) \to \beta/\sqrt{w_\star}$ as $z \to \infty$. These jointly imply $c \to 0$ and $c \geq 0$ eventually (a non-increasing c converging to 0 stays non-negative). The skew $s(z) = \partial_z \psi(z)$ is then eventually non-decreasing, and for any z_b past this threshold,

$$\int_{z_b}^{\infty} c(z')\,\mathrm{d}z' \;=\; \beta/\sqrt{w_\star} - s(z_b) \;<\; \infty. \tag{24}$$

Since $s' = c \geq 0$ on $[z_b, \infty[$, the left-hand side equals the total variation of s on $[z_b, \infty[$, so the skew has *finite total variation on the wing*. From (24) and eventual monotonicity of c, the elementary bound $(z/2)\,c(z) \leq \int_{z/2}^{z} c(z')\,\mathrm{d}z' \to 0$ gives $c(z) = o(1/z)$ as $z \to \infty$. Combined with (23), $c(z)/c_{\text{cross}}(z) \to 0$, so $c(z) < c_{\text{cross}}(z)$ holds asymptotically with vanishing-but-positive margin. Butterfly-binding holds in the deep wing under total-variation-finite skew with eventually monotone c; the Mills-ratio bound is not binding in the deep wing.

SVI and SSVI convexity decay. (Notation: from this paragraph onward σ denotes the SVI/SSVI parameter entering the wing formula below, not the Black-Scholes implied-volatility curve $\sigma(k)$ of §1.) We translate to CVI moneyness by fixing any anchor $w_\star > 0$ (e.g., $w_\star = w(0)$, the ATM total variance of the fit) and setting $z = k/\sqrt{w_\star}$, $\psi(z) = w(k)/w_\star$; this is a linear change of variable, so $c(z) = \partial_z^2 \psi = \partial_k^2 w$ exactly. For raw SVI $w(k) = a + b\rho(k-m) + b\sqrt{(k-m)^2 + \sigma^2}$, $\partial_k^2 w = b\sigma^2/((k-m)^2 + \sigma^2)^{3/2} \sim b\sigma^2/k^3$ as $k \to \infty$, hence $c_{\text{SVI}}(z) = O(1/z^3)$. The same square-root structure in SSVI $w(k,T) = (\theta/2)(1 + \rho\phi k + \sqrt{(\phi k + \rho)^2 + 1 - \rho^2})$ gives $\partial_k^2 w = \theta\phi^2(1-\rho^2)/(2\,((\phi k+\rho)^2+1-\rho^2)^{3/2})$, hence $c_{\text{SSVI}}(z) = O(1/z^3)$. In both parametrizations the second derivative is *strictly positive everywhere* (under generic $b, \sigma > 0$; $\theta, \phi > 0$, $|\rho| < 1$) and *monotonically decreasing* in $|k-m|$ (resp. $|\phi k + \rho|$), satisfying hypothesis (i) of the previous paragraph. Combined with the asymptotically linear wing of SVI and SSVI (hypothesis (ii), mentioned in this section's setup), SVI and SSVI inherit asymptotic butterfly-binding from the previous paragraph. The stronger $c(z) = O(1/z^3)$ rate is the key input for the at-and-beyond single-strike check below.

Asymptotic single-point check for SVI/SSVI butterfly arbitrage.
Proposition 10.1 (Single-strike at-and-beyond check). *For an implied-volatility parametrization with asymptotic linear-in-variance wing ($s(z) \to \beta/\sqrt{w_\star}$ as $z \to \infty$, $0 < \beta < 2$, with wing intercept w_0 in $w(k) \sim \beta k + w_0$ satisfying $\beta \neq w_0$), $c(z) = o(1/z^2)$ in the deep wing, and $c(z) \geq 0$ eventually, under Black-Scholes pricing there is a parametrization-dependent threshold $z_\star$ such that for any wing strike $z_b \geq z_\star$:*

$$s(z_b) \leq s_-(z_b, 0) \implies \textit{strike-arbitrage-free for all } z \geq z_b.$$

(Mirror at the left wing.)

Proof. The gap-monotonicity argument below uses the hypothesis $c(z) = o(1/z^2)$ (strictly stronger than the $o(1/z)$ inherited from the butterfly-binding result above; comfortably satisfied by SVI/SSVI's $O(1/z^3)$ rate from the convexity-decay paragraph above). Including the wing intercept w_0 via the "Independence from the wing intercept" remark after (21),

$$\sqrt{w_\star}\, s_-(z,0) \;=\; \frac{4\beta}{2+\beta} \;-\; \frac{8(\beta-w_0)}{(2+\beta)^2\,k} \;+\; O(k^{-2}),$$

and differentiating along the smile (via $\partial_z = \sqrt{w_\star}\,\partial_k$; the asymptotic expansion's term-by-term differentiation is justified by assuming $s_-(z,0)$ is sufficiently smooth on the wing),

$$\partial_z s_-(z,0) \;\sim\; \frac{8(\beta-w_0)}{(2+\beta)^2\,k^2} \;=\; O(1/z^2),$$

one order of z slower than c (under the Proposition's $\beta \neq w_0$ hypothesis, so the leading $1/k^2$ term is non-zero), which gives the at-and-beyond propagation through the gap-monotonicity argument we detail below.

Indeed, define the gap $G(z) \coloneqq s_-(z,0) - s(z)$ between the maximum-skew bound and the actual skew; the at-and-beyond claim $s(z) \le s_-(z,0)$ for all $z \ge z_b$ reads $G(z) \ge 0$, to be propagated from the boundary $G(z_b) \ge 0$ given the hypothesis $s(z_b) \le s_-(z_b,0)$. G depends on the specific parametrization through $s(z)$, so no direct manifestly-non-negative expression is available; we instead analyze G via its derivative and its asymptotic limit. Its derivative is $G'(z) = \partial_z s_-(z,0) - c(z) \sim 8(\beta-w_0)/((2+\beta)^2\,k^2) - c(z)$, where $c(z) = o(1/k^2)$ is asymptotically dominated by the leading $1/k^2$ term past a parametrization-dependent threshold (for SVI specifically, $c(z) \sim b\sigma^2/k^3$ and the threshold is of order $b\sigma^2/|\beta - w_0|$; the hypothesis $\beta \neq w_0$ excludes the degenerate boundary case where the leading $1/k^2$ vanishes, which would require an $O(1/k^3)$ next-order analysis). The gap limit is $G(z) \to \beta(2-\beta)/((2+\beta)\sqrt{w_\star})$ (the §7 limit of $s_-(z,0)$, namely $4\beta/((2+\beta)\sqrt{w_\star})$, minus the wing slope $\beta/\sqrt{w_\star}$ from hypothesis (ii)); strictly positive under $\beta \in\,]0,2[$, giving a floor for G on the deep wing. The sign of the leading term in $G'(z)$ is that of $\beta - w_0$, and the at-and-beyond propagation $G(z) > 0$ for $z > z_b$ (with z_b past the threshold) follows by two distinct mechanisms:

- $\beta > w_0$: G is monotone non-decreasing past the threshold (in fact strictly increasing at leading order), approaching the positive limit from below; the boundary check $G(z_b) \ge 0$ propagates by *monotonicity*, giving $G(z) > G(z_b) \ge 0$ for $z > z_b$.
- $\beta < w_0$: G is monotone non-increasing past the threshold but bounded below by the *positive limit* (approached from above); $G(z) > 0$ holds automatically past the threshold, so the boundary check is trivially satisfied.

With $c \ge 0$ eventually, the practical sufficient condition of §8 ($s \le s_-(z,0)$ implies $s \le s_-(z,c)$, since s_- is monotone in c) propagates the boundary check $s(z_b) \le s_-(z_b,0)$ to $s(z) \le s_-(z,c)$ for all $z \ge z_b$, hence butterfly admissibility throughout the wing past z_b. Black-Scholes pricing closes the chain (Roper, §5: butterfly + Lee tail $\Longrightarrow$ strike-arbitrage-freeness, via Roper's Proposition 2.11(1) supplying $d_+ \to -\infty$ from the call-side strict Lee tail). The mirror at the left wing follows by $k \to -k$ symmetry, with the put-side $d_- \to +\infty$ given by the algebraic argument of Roper's Proposition 2.11(1) applied to $d_-(x,\Xi) = -x/\Xi - \Xi/2$ under the put-side strict Lee tail $\limsup_{x\to-\infty} \Xi(x)/\sqrt{2|x|} < 1$. □

This is the SVI/SSVI analogue of the CVI "at-and-beyond" equivalence of Proposition 7.1: a boundary check propagates strike arbitrage-freeness to the entire wing past z_b (butterfly via the Proposition above; vertical-spread by §5's Roper chain, with strict Lee already a standing assumption of this section). The implication is one-way here as a propagation statement (we use the surrogate $s_-(z,0)$ rather than the exact $s_-(z,c)$ of §8); pointwise at the boundary the §8 form is necessary-and-sufficient, and the surrogate slack $s_-(z,c) - s_-(z,0) = O(c) = O(1/k^3)$ for SVI/SSVI keeps the sufficient check asymptotically tight. CVI escapes the issue because $c \equiv 0$ exactly past the edge knot, so the surrogate *is* the exact bound. The linear extrapolation also promotes the strict-finite-knot Lee bound to strict Lee throughout the wing, whereas SVI/SSVI need strict Lee as a separate parametric input ($b(1+\rho) < 2$ for SVI; analogous for SSVI).

The check sits on a different axis from the existing SVI/SSVI no-arb literature. Parametric global conditions on the parameter vector are given by [2] for raw SVI (necessary-and-sufficient, their Theorem 6.2) and [5] for SSVI (sufficient on (θ, ϕ, ρ), their Theorem 4.2); asymptotic slope (not butterfly) bounds by [3] and [10]; numerical enforcement of the Durrleman positivity condition $g(k) \geq 0$ (Durrleman [11], formalized in Roper [9]) during calibration appears in robust calibration work such as [12]. Our check is *single-strike* with *at-and-beyond* propagation: given a single wing strike k_b, the inequality of Proposition 10.1 in $w(k_b)$ and $w'(k_b)$ guarantees strike arbitrage-freeness on $[k_b, \infty[$ via the propagation argument above. To our knowledge no comparable single-strike butterfly check with proven at-and-beyond propagation has been previously formulated for SVI/SSVI.

Explicit check for raw SVI. (Reminder: σ here is the SVI parameter, cf. the notation note above.) Substituting raw SVI $w(k) = a + b\rho(k-m) + b\sqrt{(k-m)^2 + \sigma^2}$ into the regular-form butterfly roots of §6:

$$\sqrt{w_\star}\, s_\pm(z_b, 0) \;=\; \frac{4\,w(k_b)}{2k_b \mp \sqrt{w(k_b)\big(w(k_b)+4\big)}}, \tag{25}$$

and the SVI slope at k_b reads

$$\sqrt{w_\star}\, s(z_b) \;=\; \partial_k w(k_b) \;=\; b\rho + \frac{b(k_b - m)}{\sqrt{(k_b-m)^2 + \sigma^2}}. \tag{26}$$

The right-wing check of Proposition 10.1 writes explicitly, with $w_b := w(k_b)$:

$$\boxed{\, b\rho + \frac{b(k_b - m)}{\sqrt{(k_b-m)^2 + \sigma^2}} \;\leq\; \frac{4\,w_b}{2k_b + \sqrt{w_b(w_b+4)}} \,}\,. \tag{27}$$

The mirror at the left wing uses the upper (plus) root of (25), which flips the sign of the square-root term in the denominator. The SSVI analogue follows by substituting $w(k) = (\theta/2)(1 + \rho\phi k + \sqrt{(\phi k + \rho)^2 + 1 - \rho^2})$ into the same regular-form expressions.

The $\boldsymbol{\beta = w_0}$ boundary case for SVI/SSVI. The general Proposition's $\beta \neq w_0$ hypothesis is not needed for SVI/SSVI: at the boundary $\beta = w_0$ the wing's explicit next-order expansion gives $\partial_z s_-(z,0) \sim -16(\alpha+1)/((2+\beta)^2 k^3)$, where α is the $1/k$ coefficient of the wing variance ($\alpha = b\sigma^2/2$ for SVI, $\alpha = \theta(1-\rho^2)/(4\phi)$ for SSVI; both strictly positive). With $c(z) \sim O(1/k^3) > 0$ at the boundary as well, $G'(z) < 0$ past the threshold (G monotone decreasing), but the gap limit $\beta(2-\beta)/((2+\beta)\sqrt{w_\star}) > 0$ persists, so the propagation goes through via the same mechanism as the $\beta < w_0$ case (G bounded below by its positive limit). The explicit check (27) therefore applies unconditionally on SVI / SSVI.

11 Conclusion

How steep can the wing of an implied-vol smile become before strike arbitrage arises? We have addressed this in two ways: *pointwise* at any wing strike, and *at and beyond* the strike.

Both rest on the no-butterfly-arbitrage condition viewed as a quadratic in the skew, whose smaller-magnitude root is the *maximum-skew constraint* of the title (cap $s \le s_-$ on the right; mirror $s \ge s_+$ on the left). For an implied-vol parametrization (with Black-Scholes pricing), the constraint is pointwise necessary and sufficient for strike-arbitrage-freeness in the wing ($d_1 d_2 > 1$). The at-and-beyond extension holds exactly on CVI's linear wing ($c \equiv 0$ past the edge knot); more generally, the $c = 0$ form of the constraint serves as a sufficient single-strike check (under positive convexity) propagating strike arbitrage-freeness along the entire wing past the strike, in many parametrizations covering SVI and SSVI, which converge quickly to their linear asymptote.

The big picture. Theorem 9.2 consolidates the analysis: no-butterfly-arbitrage in the wings is best understood as a cap on the skew (the maximum-skew constraint), not a floor on the implied PDF (or on the convexity); the same framing extends as a band on the skew in the interior and a floor on the mirror wing. The skew is the natural variable for wing analysis: PDF and convexity vanish as $k \to \pm\infty$, while the skew generally approaches a non-zero limit. The skew cap also uses more information than the PDF floor: vertical-spread arbitrage excludes the larger-magnitude root of the butterfly quadratic, leaving the smaller-magnitude root as the binding cap; the PDF floor doesn't make that distinction. At sufficiently concave interior points the butterfly quadratic has no real roots and the constraint reverts to a convexity floor $c \ge c^*$, recovering the classical ATM PDF positivity $c \ge -2$. The mirror exception sits in the wing: where $c > c^*$ the butterfly roots disappear and butterfly stops constraining the skew. In short, across the smile butterfly arbitrage is a cap or floor on the skew, except where convexity takes over: the W-shape interior and the high-convexity wing.

Linear-in-variance wing extrapolation ($c = 0$). At the edge knot of a CVI cubic spline [1], linear-in-variance wing extrapolation forces $c = 0$, and in the regime $d_1 d_2 > 1$ the quadratic has two real roots $s_- < s_+$. The CVI paper selected the smaller-magnitude root on practical grounds (the retained root already implies a steep skew at the limit of strike arbitrage; the opposite root would be steeper still), leaving open whether any strike-arbitrage-free smile could lie on the larger-magnitude root. The present note shows it cannot, via Lucic's pointwise strike-arbitrage slope bound; the argument applies to any implied-volatility parametrization with linear-in-variance wing extrapolation.

The proof rests on the bound $s_L^{\text{right}} < s_+$ at the right edge (§4, with the $k \to -k$ mirror $s_L^{\text{left}} > s_-$ at the left), derived by direct algebra from Lucic's bound [6] and the roots (3); this rules out the larger-magnitude root from any strike-arbitrage-free smile (§5).

We then strengthen the larger-magnitude-root exclusion by casting the maximum-skew constraint as a *finite-strike form of Lee's asymptotic large-strike bound* at the edge knot (§7). Under linear-in-variance wing extrapolation, the constraint reads $s \le s_-$ at the right edge knot ($s \ge s_+$ at the left, by mirror); it guarantees the finite-strike form of Lee's bound at the knot and butterfly admissibility at and beyond the knot. Two pieces:

- *Lee's bound at the knot* (large-strike at the right, small-strike at the left): strictly tighter than Lee at any finite knot, approaching Lee only as the knot moves deep into the tail.
- *Tail propagation of butterfly admissibility*: any admissible s at the knot extends self-consistently along the entire linear tail, since s_- approaches its asymptotic limit from below at the right (mirror from-above for s_+ at the left). This asymptotic monotonicity was stated without proof in [1]; it is derived in §7. Butterfly admissibility then propagates from the knot to the whole wing without separate enforcement.

The larger-magnitude-root exclusion clarifies the role of vertical-spread arbitrage in CVI calibration. CVI enforces butterfly admissibility and Lee tail control at the edge but no direct no-vertical-spread constraint; under strict Lee and Black-Scholes pricing each CVI slice is butterfly-arbitrage-free, hence strike-arbitrage-free (§5), and (9) forces every calibrated smile onto the smaller-magnitude branch. Vertical-spread arbitrage is therefore impossible in calibrated CVI smiles by construction; no separate vertical-spread constraint is needed.

The constraint also admits an equivalent expression free of the 0/0 pathology at the regime boundary $d_1 d_2 = 1$ (§6; the edge-knot slope boundary of [1]); §9 maps the root trajectories across the smile, with one branch escaping through infinity at each regime boundary, reconciling how the smaller-magnitude root is selected at each edge (cap $s \le s_-$ on the right; floor $s \ge s_+$ on the left). With the CVI $c = 0$ case settled, the remainder of the note relaxes this assumption.

Generalization to $c \neq 0$. All preceding results rest on the $c = 0$ assumption at the edge knot, which §8 relaxes. Strict Lee gives the wing regime asymptotically; for asymptotically linear smooth tails with $c(z) \to 0$, including SVI/SSVI, the remaining real-root condition $c \le c^*(z)$ also holds asymptotically in either wing. The larger-magnitude root is still excluded wherever such roots exist, generalizing the linear-wing result. Hence, under Black-Scholes pricing, $s \le s_-$ alone is pointwise necessary and sufficient for strike-arbitrage-freeness at any right-wing point with $d_1 d_2 > 1$ (mirror at the left). The name *maximum-skew constraint* thus becomes literal: in either wing, wherever the smaller-magnitude root exists, it is the maximum admissible skew. Without Black-Scholes pricing, the combined butterfly + Mills form $s \le \min\{s_-,\ s_{\mathrm{VS}}^{\mathrm{call}}\}$ takes over.

Interplay of butterfly and vertical-spread arbitrage. All wing-relevant constraints are caps on the skew: Lee's tail-slope bound, the vertical-spread Mills threshold, and the butterfly maximum-skew constraint. Three threads connect the §8 pointwise picture to the §10 wing-asymptotic analysis.

- On the linear wing, the butterfly threshold $\sqrt{w_\star}\, s_-(z, 0)$ and the Mills threshold $\sqrt{w_\star}\, s_{\mathrm{VS}}^{\mathrm{call}}(z)$ share the same Lee saturation limit $4\beta/(2+\beta) < 2$ (strict under strict Lee), with an $O(1/k)$ gap; both approach Lee's bound $\sqrt{w_\star}\, s < 2$ only in the simultaneous limit $\beta \to 2$ and deep wing.
- The transition convexity $c_{\mathrm{cross}}(z)$, at which butterfly and Mills coincide, decays as $1/z$ along the wing (eq. (23)); the butterfly-binding window shrinks, but $c(z)$ decays faster for regular smiles (eventually monotone c on the wing), so butterfly remains the binding constraint asymptotically.
- Gordon's bound forces $c_{\mathrm{cross}} < c^*$, so the Mills threshold continuously takes over before the butterfly root disappears, keeping the admissible set of skews continuous in c at c^* given the larger-magnitude-root exclusion (also implied by Gordon).

Beyond the wing (§9), both butterfly and vertical-spread extend pointwise to the entire smile in parallel: butterfly as cap on the right wing, band $s_+ \le s \le s_-$ in the interior, floor on the left wing; vertical-spread as a pointwise two-sided bound (call-spread cap, put-spread floor) in all regimes. The one caveat is at sufficiently concave interior points where the butterfly band collapses (no real roots) and butterfly imposes a convexity floor $c \ge c^*$ instead of a skew bound. So butterfly, and strike arbitrage in general, is *mostly* about the skew.

Wing-asymptotic refinement and SVI / SSVI single-strike check. The $c = 0$ surrogate s_-^{CVI} (the $c = 0$ value at z, evaluable without estimating c) is a sufficient pointwise bound under $c \ge 0$, since $s_-^{\mathrm{CVI}} \le s_-(z, c)$ (§8). Combining §7's exact CVI propagation with §10's asymptotic extension, the at-and-beyond $c = 0$ single-strike check propagates strike arbitrage-freeness past any wing strike in three cases:

- for CVI's exact linear wing by §7, with s_-^{CVI} coinciding with the exact $s_-(z,c)$ (necessary and sufficient);
- for SVI and SSVI (where $c(z) = O(1/z^3)$) by Proposition 10.1 (sufficient), with the closed-form check worked out for raw SVI in eq. (27);
- more generally, for any parametrization with asymptotic linear-in-variance wing, $c(z) \geq 0$ eventually, and $c(z) = o(1/z^2)$ (sufficient).

Strict Lee enters as a parametric input for SVI/SSVI ($b(1+\rho) < 2$ for SVI; analogous for SSVI), whereas in the CVI case it is inherited from the boundary check itself (linear extrapolation promotes the strict-finite-knot Lee bound to strict Lee throughout the wing).

Open questions. Two directions remain open. (1) Extending the pointwise necessary-and-sufficient form of §8 to a full at-and-beyond necessary-and-sufficient form at general c: §10 establishes the asymptotic sufficient case (under $c(z) = o(1/z^2)$ and $c(z) \geq 0$ eventually), but the corresponding necessary-and-sufficient form for arbitrary $c(z)$ remains future work. (2) An SVI-specific single-strike necessary-and-sufficient analog of Proposition 10.1. The present check is sufficient and is only an asymptotic statement; the corresponding necessary-and-sufficient form is open. A candidate route is to adapt the Martini-Mingone parametric necessary-and-sufficient framework [2] to a single-strike form.

The black holes metaphor. Tracking $s_+(z)$ along the right half of the smile (Figure 4) reveals the mechanism. In the interior, s_+ is the lower band-edge root; as z moves toward the right regime boundary z_{up}, s_+ descends and asymptotes to $-\infty$ at z_{up}, crossing the vertical-spread put bound $s_{\mathrm{VS}}^{\mathrm{put}}$ on the way down. Past z_{up}, s_+ re-emerges from $+\infty$ on the wing side as the larger-magnitude positive root, above the vertical-spread call bound $s_{\mathrm{VS}}^{\mathrm{call}}$. Vertical-spread arbitrage on each side of the singularity at $d_1 d_2 = 1$ acts as an event horizon: the singularity is plainly present in the equations but hidden from us.

The metaphor is incomplete. As we approach a real black hole's horizon, escape grows ever harder, and past it, falling into the singularity becomes inescapable. For us living in an arbitrage-free world, the dynamics reverse: as we approach a vertical-spread horizon, the pathology beyond it pushes us away. The singularity at $d_1 d_2 = 1$ is purely algebraic, and crossing the horizon would constitute arbitrage. Black holes hide a real singularity at the core; Black-Scholes has the horizon and no core.

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