Convex Volatility Interpolation

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VOL*ptima*

Approaches to Vol Fitting

- **Follow a model**
  - Parsimonious by design: a handful of parameters per expiry against 100+ listed strikes $\rightarrow$ potentially **under-parameterized**.
  - Parameters enter nonlinearly $\rightarrow$ **non-convex** calibration.

  $\rightarrow$ We learn from models, but do not apply them directly to vol fitting.

- **Don't follow a model**
  - More flexibility, but **nothing does the work for you** $\rightarrow$ the calibration carries the burden.
  - Can use convex optimization (not automatic, but now available), and then afford to be **over-parameterized**.
  - **Modular**: encode vol surface properties directly in the calibration.

$\rightarrow$ **CVI sits in the second category.**

Today: the *Risk* paper. Some comments on running it in production on slide 30.

Quadratic Programming in Variance

Under the radar until recently. Why?

- **Practical frictions**
  - Before **Clarabel** (2022), an open-source solver: buy a commercial licence, or live with slower, less reliable convergence.
  - **Complexity**: the calibration is low-level, properties encoded one at a time.
- **A misconception about splines**
  - "Splines oscillate." They do, unless the fit is regularized and constrained.
- **A gap in the literature**
  - QP in **price** space is well covered; **variance** space was not.
  - The **linearly extrapolated wing** was left untreated.

Why Variance Space?

- Absence of static arbitrage = no calendar-spread + no strike arbitrage.
    - **No-calendar-spread arbitrage:** linear in both price and variance.
    - **No-butterfly arbitrage:** linear in price, but not in variance.
- **Price-space drawbacks:**
    - Wing instability (OTM prices $\to 0$, small absolute errors $\to$ large vol errors).
    - No intuitive parameters on prices.

$\to$ **Variance space is worth the trade-off:**

- Calendar constraints are still linear.
- Butterfly constraints become nonlinear $\to$ we linearize them.
- Intuitive parameters (skew, convexity) are natural in variance space.

Using Basis Functions to Leverage Convex Optimization

- Objective: least squares on the **variance at each listed strike** $\rightarrow$ quadratic in $v(K_j)$.
- Quadratic *in the optimization variables* requires v to be **linear in its parameters**:

$$v(K) = \sum_i w_i \, \phi_i(K)$$

- This restricts us to a space of **basis functions**.
- **Prior work in price space:**
  - Fengler (2009): constrained natural cubic spline smoothing of call prices.
  - Lucic (2019): B-splines with bid-ask bounds via QP.

$\rightarrow$ Similar idea here, but in **variance space**.

Basis Functions

- Choice: **degree 3 B-splines** (the basis functions of cubic splines) in **normalized log-moneyness** z.
- Linear extrapolation in the wings.

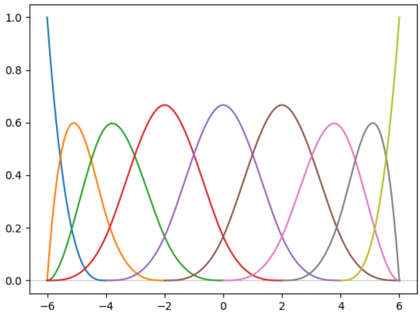


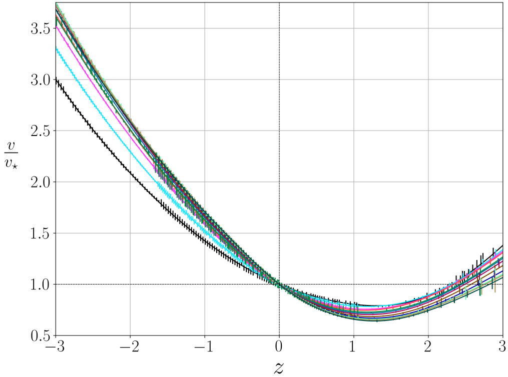
Figure: The 9 degree-3 B-spline functions associated with the 7 knots $[-6, -4, -2, 0, 2, 4, 6]$.

Notation

- $k := \log(K/F)$ log-forward moneyness
- $z := \dfrac{k}{\sigma_\star\sqrt{T}}$ normalized log-moneyness
- $\sigma_\star$ anchor ATM volatility (estimated prior to fitting)
- $v(z, T) := \sigma^2(z, T)$ variance; vT total variance
- Shape function: $z \mapsto \dfrac{v(z)}{v_\star}$ (variance normalized by $v_\star := \sigma_\star^2$)
- $s := \dfrac{1}{v_\star}\dfrac{\partial v}{\partial z}$ dimensionless skew
- $c := \dfrac{1}{v_\star}\dfrac{\partial^2 v}{\partial z^2}$ dimensionless convexity
- $d_{1,2} := \dfrac{-k \pm \frac{vT}{2}}{\sqrt{vT}}$ the Black-Scholes d_1, d_2
- m expiries: $T_1, \ldots, T_m$

The Shape Function

- $z \mapsto v(z)/v_\star$: variance normalized by anchor ATM variance.
- Approximately the same shape across all expiries.
- Allows comparing smile shapes independent of ATM level and time.



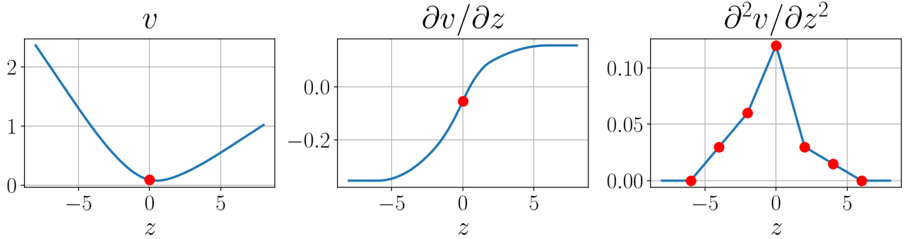
Shape function $v(z)/v_\star$ across expiries (S&P 500).

The CVI Cubic Spline

The cubic spline has $n+2$ parameters per expiry:

- $v(z=0)$: the ATM variance (1 parameter)
- $s(0) = \frac{1}{v_\star}\frac{\partial v}{\partial z}\Big|_{z=0}$: the ATM skew (1 parameter)
- $\left\{ c(z_i) = \frac{1}{v_\star}\frac{\partial^2 v}{\partial z^2}\Big|_{z_i} \right\}_{0\le i\le n-1}$: the n convexities at knots $z_0, \ldots, z_{n-1}$

Edge convexities $c(z_0) = 0$ and $c(z_{n-1}) = 0 \to$ linear extrapolation in tails.



CVI cubic spline (left), first derivative (middle), second derivative (right). Red dots = parameters.

Dual Parameterization

The same variance function can be expressed as:

- **Cubic spline parameters** (intuitive):
  - ATM variance, ATM skew, convexities at knots
- **B-spline weights**:
  - $v(z) = \sum_i w_i B_i(z) \quad \text{for } z \in [z_0, z_{n-1}]$
  - Linear extrapolation beyond edge knots

Linked by a linear map $\mathbf{w} = M\,\mathbf{p}$, known in advance ($\sigma_\star$ and the knots).

Objective Function

Four penalty terms, each scaling as a **chi-squared statistic**:

1. **Least Squares Penalty**: fit to the variance implied by the mid price.
2. **Above Ask Penalty**: penalize fitting above the ask.
3. **Below Bid Penalty**: penalize fitting below the bid.
4. **Strike Regularization**: smoothness across strikes.

- All expiries fitted **simultaneously**: one global convex optimization.
- Each penalty is the sum of **dimensionless** numbers across expiries.
- Consistent behaviour regardless of the number of expiries, options per expiry, or volatility level.

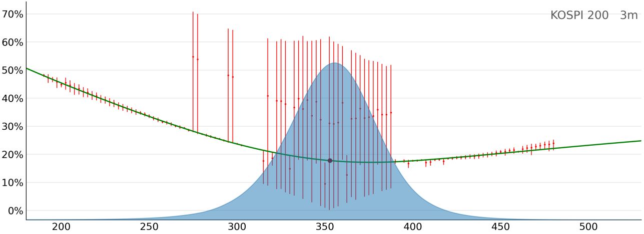
Least Squares Penalty

$$\sum_{j=1}^{m} \frac{1}{N_{T_j,\mathrm{mid}}} \sum_{i \in \mathcal{S}_{T_j,\mathrm{mid}}} \left(v(K_i, T_j) - v_{\mathrm{mid}}(K_i, T_j) \right)^2 w_{i,j}$$

Weights inversely proportional to the squared bid-ask spread in variance:

$$w_{i,j} = \frac{1}{\left(v_{\mathrm{ask}}(K_i, T_j) - v_{\mathrm{bid}}(K_i, T_j) \right)^2}$$

$\rightarrow$ Exact χ^2: tight quotes weigh more, wide quotes negligible.



KOSPI 200: wide quotes around ATM do not distort the fit.

Above Ask and Below Bid Penalties

Above Ask: penalizes $\max(v(K_i, T_j) - v_{\text{ask}}(K_i, T_j), 0)^2$

- Weights proportional to vega:
  $$w_{i,j,\text{ask}} = q_j \frac{\mathrm{Vega}_{\text{ask}}(K_i, T_j)}{\sum_{l \in \mathcal{S}_{T_j,\text{ask}}} \mathrm{Vega}_{\text{ask}}(K_l, T_j)}$$
- q_j scales the penalty to be consistent with the chi-squared of the least squares term.

Below Bid: symmetric (mirrors Above Ask).

Both terms:

- Capture information from **one-sided quotes** (especially OTM wings).
- Trade-off: require auxiliary variables, increase problem size.
- Practical choice: apply only to strikes with bid or ask, but not both.

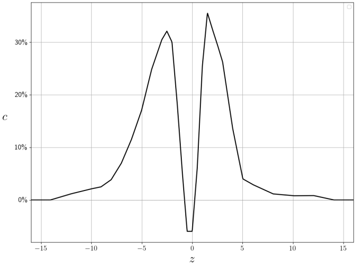
Strike Regularization Penalty

Total variation of the convexity curve:

$$\lambda \sum_{j=1}^{m} \sum_{i=0}^{n-2} \left| \mathbf{c}_{\mathbf{T}_j}[i+1] - \mathbf{c}_{\mathbf{T}_j}[i] \right|$$

where $\mathbf{c}_{\mathbf{T}_j} = \big(c(z_0, T_j), \ldots, c(z_{n-1}, T_j)\big)$.

- $\lambda = 0.05$: sensible default across underlyings.
- Interpretation: increase χ^2 by 0.05 to reduce total variation by 1.
- The c curve can exhibit well-defined features (peaks, troughs).



Positivity and Extrapolation Constraints

- **Linear Extrapolation:**
  - $c(z_0) = 0$ and $c(z_{n-1}) = 0$
  - Variance linearly extrapolated in log-strike beyond edge knots.
- **Positivity of the Variance inside the Edge Knots:**
  - Impose **B-spline coefficients** $\geq 0 \rightarrow v(z) \geq 0$ everywhere.
  - Only for the first expiry (subsequent bounded by calendar constraints).
- **Positivity of the Variance in the Tails:**
  - $\frac{\partial v}{\partial z}(z_0) \leq 0 \quad$ (non-increasing in left tail)
  - $\frac{\partial v}{\partial z}(z_{n-1}) \geq 0 \quad$ (non-decreasing in right tail)

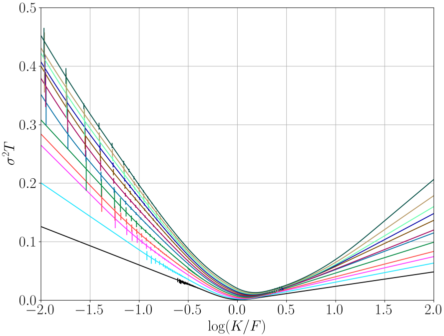
All constraints above are **linear**.

No-Calendar-Spread-Arbitrage Constraints

Total variance must increase with maturity for a fixed strike-to-forward ratio:

$$v(K_i, T_j)\, T_j \;\leq\; v\left(K_i \frac{F_{T_{j+1}}}{F_{T_j}},\; T_{j+1}\right) T_{j+1}$$

- Applied to r strikes: $(m-1)r$ **linear** constraints.
- Total variance curves must not intersect.



Calendar Constraints in the Tails

Beyond edge knots, variance is linearly extrapolated. Differentiate the calendar constraint w.r.t. $\log K$:

$$\begin{cases} s(z_0, T_j)\sqrt{v_\star(T_j)T_j} & \geq s(z_0, T_{j+1})\sqrt{v_\star(T_{j+1})T_{j+1}} \\ s(z_{n-1}, T_j)\sqrt{v_\star(T_j)T_j} & \leq s(z_{n-1}, T_{j+1})\sqrt{v_\star(T_{j+1})T_{j+1}} \end{cases}$$

- Equivalently: $T\dfrac{dv}{dk}$ is non-increasing at z_0 and non-decreasing at z_{n-1}.
- Only $2(m-1)$ additional **linear** constraints.

No-Butterfly-Arbitrage: The Challenge

From Martini and Mingone (2022): $\sigma(k)\, d_1'(k)\, d_2'(k) + \sigma''(k) \geq 0$.

Change of variables to normalized log-moneyness z:

$$c \geq \frac{v_\star}{2v} s^2 + \sqrt{v_\star T}\, s - 2\left(1 + d_1 \sqrt{\frac{v_\star}{v}}\, s + d_1 d_2 \frac{v_\star}{4v} s^2\right)$$

where $d_{1,2} = \dfrac{-k \pm \frac{vT}{2}}{\sqrt{vT}}$.

- This constraint is **nonlinear and nonconvex** in variance space.
- Cannot be enforced directly in a QP.

Linearization of Butterfly Constraints

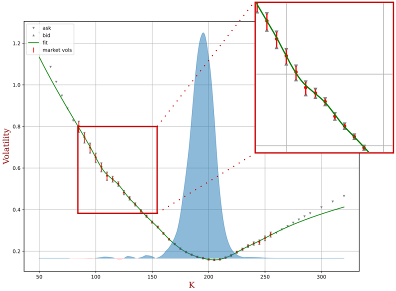
Linearize around the reference solution (previous iteration):

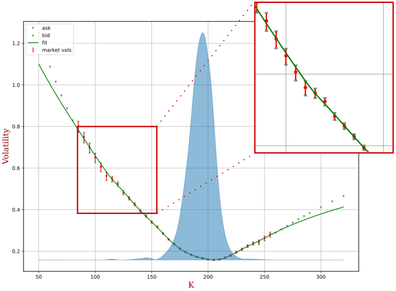
$$c \geq \alpha_0 + \alpha_1(s - s_{\mathrm{ref}}) + \alpha_2(v - v_{\mathrm{ref}})$$

$$\begin{cases} \alpha_0 = & \frac{v_\star}{2v_{\mathrm{ref}}} s_{\mathrm{ref}}^2 + \sqrt{v_\star T}\, s_{\mathrm{ref}} - 2\left(1 + d_{1,\mathrm{ref}} \sqrt{\frac{v_\star}{v_{\mathrm{ref}}}}\, s_{\mathrm{ref}} + d_{1,\mathrm{ref}} d_{2,\mathrm{ref}}\, s_{\mathrm{ref}}^2 \frac{v_\star}{4v_{\mathrm{ref}}}\right) \\ \alpha_1 = & \frac{v_\star}{v_{\mathrm{ref}}} s_{\mathrm{ref}} + \sqrt{v_\star T} - 2\left(d_{1,\mathrm{ref}} \sqrt{\frac{v_\star}{v_{\mathrm{ref}}}} + d_{1,\mathrm{ref}} d_{2,\mathrm{ref}}\, s_{\mathrm{ref}} \frac{v_\star}{2v_{\mathrm{ref}}}\right) \\ \alpha_2 = & \frac{s_{\mathrm{ref}}}{v_{\mathrm{ref}}} \left(-\frac{v_\star}{2v_{\mathrm{ref}}} s_{\mathrm{ref}} - 2\frac{k}{v_{\mathrm{ref}} T}\left(\sqrt{v_\star T} - \frac{k}{2}\, s_{\mathrm{ref}} \frac{v_\star}{v_{\mathrm{ref}}}\right)\right) \end{cases}$$

- Now **linear** in $(v, s, c) \to$ compatible with QP.
- Two-step calibration: (1) without butterfly constraints, (2) with linearized constraints.

Effect of No-Butterfly-Arbitrage Constraints





Without butterfly constraints

Kinks out of the bid-ask; density < 0

With butterfly constraints

Tracks the bid-ask; density ≥ 0

AAPL 19 Jan 2024 expiry (6 Dec 2023).

Lee's Tail Slope Bounds

Lee (2004): $\beta_{R,L} := \limsup_{k \to \pm\infty} \dfrac{v(k)\,T}{|k|} \in [0,\,2].$

Linear wings $\Rightarrow$ the wing slope is constant and *equals* β, so it can be imposed at the knot: $\beta = |T\,\partial v/\partial \log K|$. We impose the **strict** form $\beta < 2$:

$$s(z_{n-1}) < \frac{2}{\sqrt{v_\star T}} \qquad s(z_0) > -\frac{2}{\sqrt{v_\star T}}$$

Why strict, and not Lee's $\beta \leq 2$:

- $\beta = 2$ is the boundary; strictness stays off it. Strict Lee + linear wings give $d_1 \to -\infty$ as $k \to +\infty$ (large-strike no-arb.) and $d_2 \to +\infty$ as $k \to -\infty$ (no mass at zero); both collapse to 0 at $\beta = 2$.
- It pushes $d_1 d_2 \to +\infty$: past $|k| > \sqrt{vT}\sqrt{1 + \frac{vT}{4}}$ the wing is in the regime $d_1 d_2 > 1$, and the edge knots sit past that.

$\rightarrow$ That regime is what the **edge-knot butterfly constraint** needs.

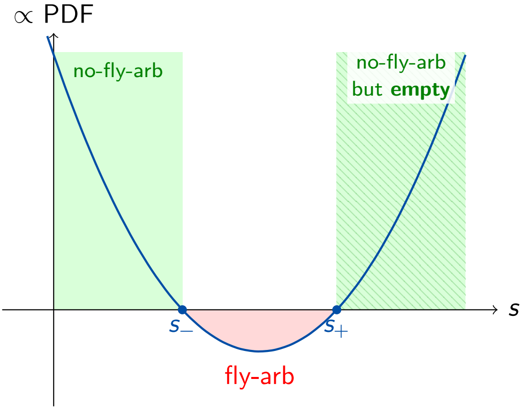
PDF ≥ 0 at Edge Knots: Quadratic in s

At edge knots, $c = 0$. The butterfly condition simplifies to a quadratic in s:

$$v_\star \left(\frac{d_1 d_2 - 1}{2v} \right) s^2 + \sqrt{v_\star} \left(\frac{2d_1}{\sqrt{v}} - \sqrt{T} \right) s + 2 \geq 0$$

- Discriminant: $\Delta = v_\star \left(T + \dfrac{4}{v} \right) > 0 \quad$ (always two real roots).
- Edge knots placed far enough into the tails so that $d_1 d_2 > 1$ (positive quadratic coefficient).

PDF ≥ 0 at Edge Knots: No-Arbitrage Region



Right edge z_{n-1}: $0 < s_- < s_+$. Left edge z_0 is the mirror ($s_- < s_+ < 0$; keep $s \geq s_+$).

$\rightarrow$ Outside both roots $\Rightarrow$ PDF ≥ 0, but only the smaller-$|s|$ branch stays **non-empty** under strike-arbitrage-freeness.

- Indeed, every skew on the larger-$|s|$ branch carries a **vertical-spread arbitrage** (CDF > 1) $\Rightarrow$ empty (`volptima.com/maximum-skew-constraint.pdf`).

Why Only at Edge Knots?

Only the smaller-$|s|$ root survives. It gives a closed-form skew cap at each edge (needs to be linearized, like the interior constraints):

- At z_{n-1} (right tail): enforce s below the lower root (positive):

$$s \leq \frac{2v\sqrt{T}\left(k - \sqrt{vT}\sqrt{1 + \frac{vT}{4}}\right)}{\sqrt{v_\star}\left(k^2 - \frac{v^2T^2}{4} - vT\right)}$$

- At z_0 (left tail): enforce s above the upper root (negative):

$$s \geq \frac{2v\sqrt{T}\left(k + \sqrt{vT}\sqrt{1 + \frac{vT}{4}}\right)}{\sqrt{v_\star}\left(k^2 - \frac{v^2T^2}{4} - vT\right)}$$

$\rightarrow$ Only need to enforce at z_0 and z_{n-1}, **not beyond**.

- Given **strict** Lee's tail slope bounds, the RHS reaches a constant asymptotically from below ($k \to +\infty$) and from above ($k \to -\infty$).

Putting It All Together: Canonical QP

CVI fits all expiries by solving a QP with linear constraints:

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2}\mathbf{x}^T P \mathbf{x} + \mathbf{q}^T \mathbf{x} \\ \text{subject to} \quad & \mathbf{l} \leq A\mathbf{x} \leq \mathbf{u} \end{aligned}$$

Constraint	Linearized	Discretized
Linear Extrapolation	–	–
Positivity of the Variance	–	✓
Positivity in the Tails	–	–
No Calendar Spread	–	✓
No Calendar Spread (Tails)	–	–
PDF ≥ 0 (interior)	✓	✓
Lee's Tail Slope Bounds	–	–
PDF ≥ 0 (edge knots)	✓	–

Two-step calibration handles the linearized constraints.

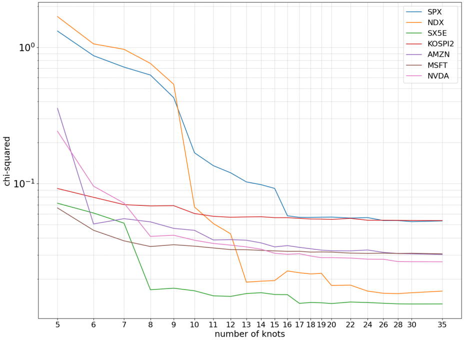
Numerical Results

	$n_{\text{vol quotes}}$	n_{expiries}	$n_{\text{constraints}}$
SX5E	3 153	19	948
SPX	14 538	46	2 298

Asset	n_{knots}	P Non-Zeros	A Non-Zeros	Time (s)
SX5E	10	1 046	14 424	0.034
	20	1 766	22 465	0.061
	40	5 836	51 710	0.155
SPX	10	2 905	32 158	0.078
	20	6 706	52 683	**0.152**
	40	30 717	122 416	0.467

- Solver: **Clarabel**, default tolerance; time totalled across 2 calibration iterations.
- SPX: $\sim$0.15 s with 20 knots $\times$ 46 expiries (920 parameters, 14 538 quotes).
- Scaling approximately linear in typical use cases.

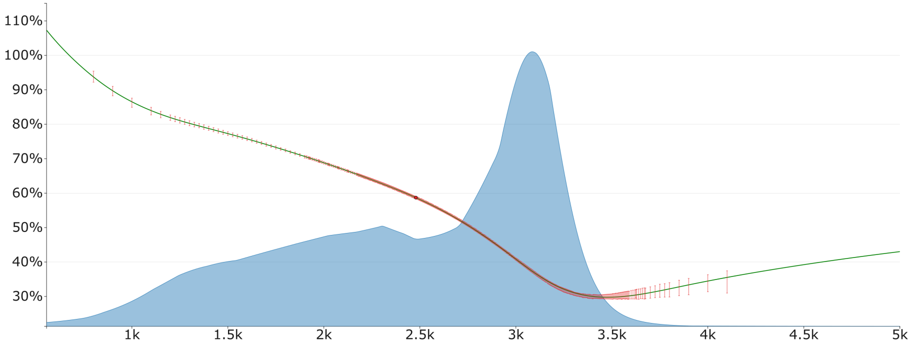
Convergence of the Fit with the Number of Knots



Chi-squared (average across expiries for one snapshot) vs. knots per expiry.

- The fit converges when the chi-squared statistic plateaus.
- Too few knots acts similarly to applying stronger regularization.
- Mid-to-high teens sufficient for highly liquid assets; 10 for less liquid ones.

Example: S&P 500 in a Stressed Market



S&P 500, 16 March 2020 $\rightarrow$ 19 June 2020 expiry ($F \approx 2479$).

More fits: `volptima.com/gallery`

Contributions of the Paper

CVI: volatility surface fitting framework via convex optimization in variance space.

Summary:

- Arbitrage-free in strike and time, including in the tails.
- All expiries fitted simultaneously.
- Hundredths to tenths of a second calibration.
- Consistent across underlyings via dimensionless parameterization.

Key contributions:

- Intuitive cubic spline parameterization with B-spline dual.
- Chi-squared-scaled objective with bid-ask awareness.
- Derived and linearized butterfly constraints in variance space.
- Treatment of strike arbitrage in the linearly extrapolated wings.

Running CVI in Production

- **What the paper gives you:**
  - Fits market data well, no hyperparameter tuning.
  - Arbitrage-free.
- **What a production system adds on top:**
  - Coupling across expiries, beyond the calendar constraint.
  - Wing stabilization.
  - Memory: a single snapshot is not enough when market makers widen quotes (sell-offs, events, ...).

→ *None of this invalidates the framework (CVI is a low-level calibration encoding a vol surface's properties). Building a production-grade system takes substantial engineering beyond the paper, which is what Volptima's fitter does.*

Follow up / Q & A

- CVI paper:
  `volptima.com/convex-volatility-interpolation.pdf`
- More about butterfly arbitrage in variance space / skew quadratic:
  `volptima.com/maximum-skew-constraint.pdf`
- Volptima's CVI fitter:
  `volptima.com/product`
- Contact:
  `fabrice.deschatres@volptima.com`

Q & A